Generalized Debye series expansion of electromagnetic plane wave scattering by an infinite multilayered cylinder at oblique incidence

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The Debye series expansion expresses the Mie scattering coefficients into a series of Fresnel coefficients and gives physical interpretation of different scattering modes, but when an infinite multilayered cylinder is obliquely illuminated by electromagnetic plane waves, the scattering process becomes very complicated because of cross polarization. Based on the relation of boundary conditions between global scattering process and local scattering processes, the generalized Debye series expansion of plane wave scattering by an infinite multilayered cylinder at oblique incidence is derived in this paper. The formula and the code are verified by the comparison of the results with that of Lorenz-Mie theory in special cases and those presented in the literatures.

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I. INTRODUCTION

Many bodies of practical interest may be considered as or closely approximated by infinite multilayered or radially inhomogeneous cylinders, for example, biological media [1], graded-index polymer optical fiber (GI-POF) [2], optofluidic wave guides [3], water-coated glass rod [4], etc. The electromagnetic scattering and absorption characteristics of these bodies are of great importance in many practical problems, such as radar cross-section studies, microwave hazards, chemical reactions, attenuation of microwave in forest, detection of objects under earth, particle sizing, optofluidic wave-guide measurement, etc. It is more commonplace that the bodies are obliquely illuminated by electromagnetic waves. The research on the interaction between infinite multilayered cylinders and oblique incident plane wave is particularly important in the study of the characteristics of such objects. But when an infinite multilayered cylinder is obliquely illuminated by electromagnetic plane waves, the scattering process becomes complicated because of cross polarization [5].

Many researchers have been devoted to the scattering of cylinders obliquely illuminated by plane waves. Wu has treated the high-frequency scattering of a perfectly conducting circular cylinder by using geometrical optics [6]. By considering the creeping waves the cross section is expressed by asymptotic expansion. Wait [7] first presented the exact solution of interaction between infinite homogeneous cylinders and oblique plane waves. In their literature, Kerker [8], van de Hulst [9], Bohren and Huffman [10] also gave the exact solution of light scattering by tilt infinite homogeneous cylinders. Barabás [11] solved the problem of inhomogeneous cylinders obliquely illuminated by plane waves, but his algorithm was limited by the size and the layers of particles. In order to solve the problem of large particles with many layers, Jiang *et al.* [2] introduced an improved algorithm. Lock [5] employed the angular spectrum of plane waves to solve the scattering of obliquely incident focused Gaussian beam by an infinitely long homogeneous circular cylinder. Ren *et al.* [12] and Mees *et al.* [13] studied the interaction between Gaussian beam and infinite homogeneous cylinders at normal and oblique incidence, and they have introduced the localized approximation to accelerate the calculation of the beam shape coefficients for cylinder. Barton [14] presented the internal and near-surface electromagnetic field for an infinite cylinder illuminated by an arbitrary focused beam.

Whereas their solutions are all complicated combinations of Bessel functions, and the mathematical complexity obscures the physical interpretation of scattering. In these researches only the total fields are given and there is no access to the contribution of different modes. The Debye series expansion (DSE) [15] is an efficient technique to make explicit the physical interpretation of different scattering modes which is implicit in the Lorenz-Mie theory. After the DSE is presented by Debye [15] in 1908 for the interaction between electromagnetic waves and cylinders, the DSE for electromagnetic scattering by homogeneous [16], coated [17], multilayered spheres [18], multilayered cylinders at normal incidence [19], homogeneous cylinder at oblique incidence [20], and spherical gratings [21] are studied. These theories have been widely employed to analyze various phenomena, such as rainbow [17,22], glories [22,23], multiple reflections and attenuation of electromagnetic plane waves in a stratified sea substratum [24], transient radiation from a Hertzian dipole antenna in a cavity in a medium [25]. However, the method used in the DSE is, in our knowledge, not applicable to the scattering of an infinite multilayered cylinder obliquely illuminated by plane waves because of cross polarization.

The generalized Debye series expansion (GDSE) employs the matrix form to expand the global scattering process in a series of local scattering processes, and has been employed to the acoustic scattering of concentric and nonconcentric multilayered cylinder [26–28] where the fields are scalar. But up to now, the GDSE has not been used for the electromagnetic scattering. Because of polarization, the process of electromagnetic scattering is more complicated than that of acoustic scattering. This paper is devoted to the development the GDSE of obliquely incident plane wave scattering by infinite multilayered cylinders.

This paper proceeds as follows: Section II derives the GDSE for homogeneous cylinder, and in Secs. III and IV, the GDSE is, respectively, extended to coated and multilayered cylinders. A numerical algorithm is presented in Sec. V. Our formula and code are verified by comparison to the presented results in special cases in Sec. VI and results of the GDSE are also presented. Section VII is devoted to the conclusions.

II. HOMOGENEOUS CYLINDER

In this section, after presenting the global scattering process, we obtain the Fresnel coefficients according to two fictitious local scattering processes: Incoming-wave scattering and outgoing-wave scattering. Finally, employing the relation between the matrix introduced by boundary conditions for the global scattering process and that for the local scattering processes, we obtain the GDSE formulation which expands the global scattering coefficients in a series of Fresnel coefficients.

In this paper, Lock's notations [5] μ and ε are used to denote the polarization states. μ-polarization corresponds to the case that the magnetic field is perpendicular to the incident plane and ε-polarization to that of the electric field is perpendicular to the incident plane. It is necessary to point out that in this paper the case for the μ-polarization is derived in detail, while the result for the ε-polarization is given directly for simplicity thinks to the similarity in the derivation for these two polarization cases.

A. Global scattering process

We consider first a homogeneous cylinder (region 1) of radius a and refractive index m_1 embedded in a medium (region 2) of refractive index m_2. The axis of the cylinder coincides with the z axis of the Cartesian coordinate system, and the incident waves are in the $x-z$ plane. A monochromatic plane wave is incident on the cylinder making an angle ξ with x axis. The time dependence of the incident waves is $\exp(-i\omega t)$. The geometry of the system is given in Fig. 1.

Suppose a μ-polarized plane wave

$$\psi_{\text{inc}}^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n J_n(x_2), \tag{1}$$

is incident on the cylinder. Then the waves in both regions 1 and 2 can be expressed as

$$\psi_1^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n C_{n,1}^{\mu} J_n(x_1), \tag{2}$$

$$\psi_1^{\varepsilon} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n D_{n,1}^{\mu} J_n(x_1), \tag{3}$$

$$\psi_2^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n [J_n(x_2) - B_{n,1}^{\mu} H_n^{(1)}(x_2)], \tag{4}$$

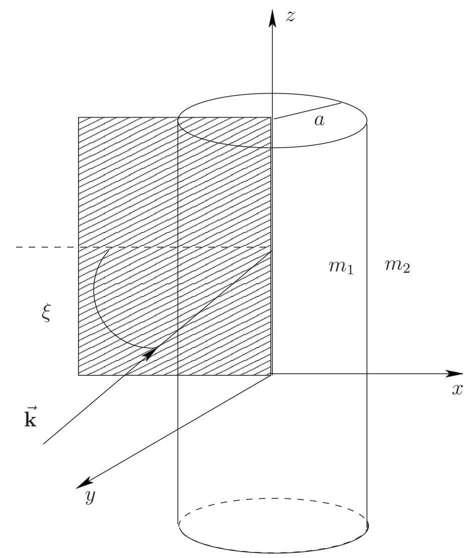


FIG. 1. Geometry of incident plane wave scattering by homogeneous cylinder at oblique incidence.

$$\psi_2^{\varepsilon} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n Q_{n,1}^{\mu} H_n^{(1)}(x_2), \tag{5}$$

where $C_{n,1}^{\mu}$ and $D_{n,1}^{\mu}$ are the internal field coefficients, and $B_{n,1}^{\mu}$ and $Q_{n,1}^{\mu}$ are the scattering field coefficients. $D_{n,1}^{\mu}$ and $Q_{n,1}^{\mu}$ are coefficients of the cross polarization. The superscript μ and the subscript 1 of all coefficients, respectively, represent the polarization states and the number of layers. The common factor $\mathcal{E}_n$ is defined by

$$\mathcal{E}_n = (-i)^n e^{-i\omega t - ihz + in\theta}. \tag{6}$$

$J_n(x)$ is the Bessel function of the first kind, $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are, respectively, the first and the second kind Hankel functions. For the sake of conciseness the following notations are used:

$$\kappa_j = \sqrt{k_j^2 - h^2}, \tag{7}$$

$$k_j = m_j k_0, \tag{8}$$

$$x_j = \kappa_j r, \tag{9}$$

$$h = k_0 \sin \xi, \tag{10}$$

where λ and $k_0 = 2\pi/\lambda$ are, respectively, the wavelength and the wave number in free space, and m_j the refractive index in region j with $j=1,2$ for homogeneous cylinder. This index j will extend to 3 for coated cylinder and to $l+1$ for l layered cylinder in the following sections.

For convenience, we define also two parameters related to the radius of each layer r_j,

$$\alpha_j = \kappa_{j+1} r_j, \tag{11}$$

$$\beta_j = \kappa_j r_j. \tag{12}$$

By using the relation

$$J_n(x) = \tfrac{1}{2}[H_n^{(2)}(x) + H_n^{(1)}(x)], \tag{13}$$

Eq. (4) can be rewritten as

$$\psi_2^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n[H_n^{(2)}(x_2) + S_{n,1}^{\mu} H_n^{(1)}(x_2)], \tag{14}$$

where

$$S_{n,1}^{\mu} = 1 - 2B_{n,1}^{\mu}. \tag{15}$$

The scattering coefficients $B_{n,1}^{\mu}$ can in return be written as

$$B_{n,1}^{\mu} = \tfrac{1}{2}(1 - S_{n,1}^{\mu}). \tag{16}$$

The first term in Eq. (16), $\frac{1}{2}$, according to the analysis of scattering coefficients by van de Hulst [9], should be the diffraction of incident waves around the particle, which is independent of the characteristics of particles. The second term corresponds to the incoming incident waves $H_n^{(2)}(k_2 r)$ which are dependent of the characteristics of the particles. Hereafter we will derive the coefficient $S_{n,1}^{\mu}$.

The electric and magnetic fields must satisfy the following boundary conditions at $r=a$, namely their tangential components must be continuous:

$$E_\theta, \quad \frac{\partial \psi^{\varepsilon}}{\partial r} + \frac{inh}{mk_0 r}\psi^{\mu} = \text{const}, \tag{17}$$

$$E_z, \quad \frac{m^2k_0^2 - h^2}{mk_0}\psi^{\mu} = \text{const}, \tag{18}$$

$$H_\theta, \quad m\frac{\partial \psi^{\mu}}{\partial r} - \frac{inh}{k_0 r}\psi^{\varepsilon} = \text{const}, \tag{19}$$

$$H_z, \quad (m^2k_0^2 - h^2)\psi^{\varepsilon} = \text{const}. \tag{20}$$

By institution of Eqs. (2), (3), (5), and (14), into Eqs. (17)–(20), and by taking into consideration of Eq. (13), we can obtain the following equation system:

$$\mathbb{D}_1 \mathbb{x}_1^{\mu} = \mathbb{h}_{\text{ext},1}^{\mu}, \tag{21}$$

where

$$\mathbb{D}_1 = \begin{bmatrix} \dfrac{inh}{k_2 a}H_n^{(1)}(\alpha) & 2\dfrac{inh}{k_1 a}J_n(\beta) & 2\kappa_1 J_n'(\beta) & \kappa_2 H_n^{(1)\prime}(\alpha) \\ \dfrac{\kappa_2^2}{m_2}H_n^{(1)}(\alpha) & 2\dfrac{\kappa_1^2}{m_1}J_n(\beta) & 0 & 0 \\ m_2\kappa_2 H_n^{(1)\prime}(\alpha) & 2m_1\kappa_1 J_n'(\beta) & -2\dfrac{inh}{k_0 a}J_n(\beta) & -\dfrac{inh}{k_0 a}H_n^{(1)}(\alpha) \\ 0 & 0 & 2\kappa_1^2 J_n(\beta) & \kappa_2^2 H_n^{(1)}(\alpha) \end{bmatrix}, \tag{22}$$

$$\mathbb{x}_1^{\mu} = \left[S_{n,1}^{\mu} \quad \tfrac{1}{2}C_{n,1}^{\mu} \quad \tfrac{1}{2}D_{n,1}^{\mu} \quad Q_{n,1}^{\mu}\right]^T, \tag{23}$$

$$\mathbb{h}_{\text{ext},1}^{\mu} = \left[\frac{inh}{k_2 a}H_n^{(2)}(\alpha) \quad \frac{\kappa_2^2}{m_2}H_n^{(2)}(\alpha) \quad m_2\kappa_2 H_n^{(2)\prime}(\alpha) \quad 0\right]^T. \tag{24}$$

To conserve the conventional notations for homogeneous cylinder we use here α and β which are related to the notations in this paper by $\alpha=\kappa_2 a=\alpha_1$ and $\beta=\kappa_1 a=\beta_1$.

The vector $\mathbb{x}_1^{\mu}$ in Eq. (21) contains scattering coefficients to be determined. $\mathbb{h}_{\text{ext},1}^{\mu}$ is the μ-polarized excitation imposed to the cylinders (external excitation). $\mathbb{D}_1$ is the specific operator introduced by the boundary conditions (global operator), and it is independent of the excitation. Generally speaking, the solution of a scattering problem in a separable geometry consists in finding the solution of such linear system obtained from the boundary conditions. The operator $\mathbb{D}_1$ being reversible, the solution can be written as

$$\mathbb{x}_1^{\mu} = \mathbb{D}_1^{-1}\mathbb{h}_{\text{ext},1}^{\mu}. \tag{25}$$

For ε-polarized wave incidence, we can obtain the similar equation:

$$\mathbb{x}_1^{\varepsilon} = \mathbb{D}_1^{-1}\mathbb{h}_{\text{ext},1}^{\varepsilon} \tag{26}$$

with

$$\mathbb{x}_1^{\varepsilon} = \left[Q_{n,1}^{\varepsilon} \quad \tfrac{1}{2}D_{n,1}^{\varepsilon} \quad \tfrac{1}{2}C_{n,1}^{\varepsilon} \quad S_{n,1}^{\varepsilon}\right]^T \tag{27}$$

and

$$\mathbb{h}_{\text{ext},1}^{\varepsilon} = \left[\kappa_2 H_n^{(2)\prime}(\alpha) \quad 0 \quad -\frac{inh}{k_0 a}H_n^{(2)}(\alpha) \quad \kappa_2^2 H_n^{(2)}(\alpha)\right]^T. \tag{28}$$

Therefore, the complete solution to the global scattering process is given by Eqs. (25) and (26). In order to develop the scattering process into Debye series we study in the following section the local scattering process.

B. Fresnel coefficients

The Fresnel coefficients are obtained by two fictitious local scattering processes corresponding to incoming and outgoing waves. The outgoing wave is related to the wave that penetrates into the cylinder and locally interacts with the internal interface. Because of the cross polarization, there are, respectively, two polarizations for outgoing and incoming waves. Therefore, in order to obtain the GDSE, we must take into consideration of four fictitious local scattering processes: μ-polarized incoming wave, μ-polarized outgoing wave, ε-polarized incoming wave, and ε-polarized outgoing wave.

1. μ-polarized incoming waves

When μ-polarized incoming wave in region 2 encounters the interface, it is partially reflected with μ polarization ($R_{\mu\mu}^{212}$) and ε polarization ($R_{\mu\varepsilon}^{212}$), and partially transmitted with μ polarization ($T_{\mu\mu}^{21}$) and ε polarization ($T_{\mu\varepsilon}^{21}$). The first and second subscripts of all Fresnel coefficients represent, respectively, the polarization states of waves before and after interaction with interface. Figure 2 depicts the geometry of such fictitious process. Then the complete waves in two regions are given by

$$\psi_1^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n T_{\mu\mu}^{21} H_n^{(2)}(x_1), \tag{29}$$

$$\psi_1^{\varepsilon} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n T_{\mu\varepsilon}^{21} H_n^{(2)}(x_1), \tag{30}$$

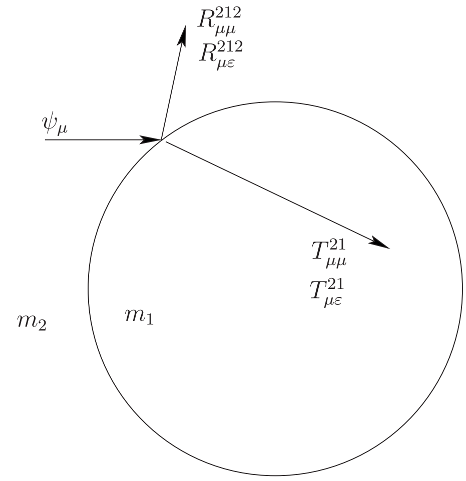


FIG. 2. Geometry of fictitious problem corresponding to μ-polarized incoming waves.

$$\psi_2^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n [H_n^{(2)}(x_2) - R_{\mu\mu}^{212} H_n^{(1)}(x_2)], \tag{31}$$

$$\psi_2^{\varepsilon} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n R_{\mu\varepsilon}^{212} H_n^{(1)}(x_2). \tag{32}$$

The boundary conditions lead to the following equation system,

$$\mathbb{D}_{L,1} \mathbb{f}_{\text{in},1}^{\mu} = \mathbb{h}_{\text{in},1}^{\mu}, \tag{33}$$

where

$$\mathbb{D}_{L,1} = \begin{bmatrix} \frac{inh}{k_2 a} H_n^{(1)}(\alpha) & \frac{inh}{k_1 a} H_n^{(2)}(\beta) & \kappa_1 H_n^{(2)\prime}(\beta) & \kappa_2 H_n^{(1)\prime}(\alpha) \\ \frac{\kappa_2^2}{m_2} H_n^{(1)}(\alpha) & \frac{\kappa_1^2}{m_1} H_n^{(2)}(\beta) & 0 & 0 \\ m_2 \kappa_2 H_n^{(1)\prime}(\alpha) & m_1 \kappa_1 H_n^{(2)\prime}(\beta) & -\frac{inh}{k_0 a} H_n^{(2)}(\beta) & -\frac{inh}{k_0 a} H_n^{(1)}(\alpha) \\ 0 & 0 & \kappa_1^2 H_n^{(2)}(\beta) & \kappa_2^2 H_n^{(1)}(\alpha) \end{bmatrix}, \tag{34}$$

$$\mathbb{f}_{\text{in},1}^{\mu} = [R_{\mu\mu}^{212} \quad T_{\mu\mu}^{21} \quad T_{\mu\varepsilon}^{21} \quad R_{\mu\varepsilon}^{212}]^T, \tag{35}$$

$$\mathbb{h}_{\text{in},1}^{\mu} = \left[\frac{inh}{k_2 a} H_n^{(2)}(\alpha) \quad \frac{\kappa_2^2}{m_2} H_n^{(2)}(\alpha) \quad m_2 \kappa_2 H_n^{(2)\prime}(\alpha) \quad 0 \right]^T. \tag{36}$$

The four Fresnel coefficients $R_{\mu\mu}^{212}$. $T_{\mu\mu}^{21}$, $T_{\mu\varepsilon}^{21}$, and $R_{\mu\varepsilon}^{212}$ can be obtained from Eq. (33) by using Cramer's rules and are given in Appendix A. It is necessary to point out that $\mathbb{D}_{L,1}$ is an operator introduced by local boundary conditions and is independent of the excitation. Therefore, $\mathbb{D}_{L,1}$ is the same for μ-polarized incoming wave, μ-polarized outgoing wave, ε-polarized incoming wave, and ε-polarized outgoing wave.

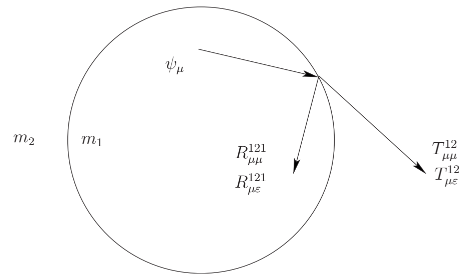


FIG. 3. Geometry of fictitious problem corresponding to μ-polarized outgoing waves.

2. μ-polarized outgoing waves

When μ-polarized outgoing wave hits the interface, it is partially reflected with μ polarization ($R^{121}_{\mu\mu}$) and ε polarization ($R^{121}_{\mu\varepsilon}$), and partially transmitted with μ polarization ($T^{12}_{\mu\mu}$) and ε polarization ($T^{12}_{\mu\varepsilon}$). Figure 3 depicts the geometry of such fictitious process. Then the complete waves in the two regions are

$$\psi_1^\mu = \sum_{n=-\infty}^{\infty} \mathcal{E}_n[H_n^{(1)}(x_1) - R^{121}_{\mu\mu}H_n^{(2)}(x_1)], \tag{37}$$

$$\psi_1^\varepsilon = \sum_{n=-\infty}^{\infty} \mathcal{E}_n R^{121}_{\mu\varepsilon} H_n^{(2)}(x_1), \tag{38}$$

$$\psi_2^\mu = \sum_{n=-\infty}^{\infty} \mathcal{E}_n T^{12}_{\mu\mu} H_n^{(1)}(x_2), \tag{39}$$

$$\psi_2^\varepsilon = \sum_{n=-\infty}^{\infty} \mathcal{E}_n T^{12}_{\mu\varepsilon} H_n^{(1)}(x_2). \tag{40}$$

The boundary conditions lead to the following system:

$$\mathbb{D}_{L,1}\mathbb{f}^\mu_{\text{out},1} = \mathbb{h}^\mu_{\text{out},1}, \tag{41}$$

where

$$\mathbb{f}^\mu_{\text{out},1} = [T^{12}_{\mu\mu} \quad R^{121}_{\mu\mu} R^{121}_{\mu\varepsilon} \quad T^{12}_{\mu\varepsilon}]^T, \tag{42}$$

$$\mathbb{h}^\mu_{\text{out},1} = \left[-\frac{inh}{k_1 a}H_n^{(1)}(\beta) \quad -\frac{\kappa_1^2}{m_1}H_n^{(1)}(\beta) \quad - m_1\kappa_1 H_n^{(1)\prime}(\beta) \quad 0 \right]^T. \tag{43}$$

All the Fresnel coefficients $T^{12}_{\mu\mu}$. $R^{121}_{\mu\mu}$. $R^{121}_{\mu\varepsilon}$, and $T^{12}_{\mu\varepsilon}$ are obtained from Eq. (41) by using Cramer's rules and listed in Appendix A.

3. ε-polarized incoming and outgoing waves

For the ε-polarized incoming and outgoing-waves scattering, the system of boundary conditions is similar to the case for μ-polarized waves and are expressed as follows.

For ε-polarized incoming waves

$$\mathbb{D}_{L,1}\mathbb{f}^\varepsilon_{\text{in},1} = \mathbb{h}^\varepsilon_{\text{in},1} \tag{44}$$

with

$$\mathbb{f}^\varepsilon_{\text{in},1} = [R^{212}_{\varepsilon\mu} \quad T^{21}_{\varepsilon\mu} \quad T^{21}_{\varepsilon\varepsilon} \quad R^{212}_{\varepsilon\varepsilon}]^T, \tag{45}$$

$$\mathbb{h}^\varepsilon_{\text{in},1} = \left[-\kappa_2 H_n^{(2)\prime}(\beta) \quad 0 \quad -\frac{inh}{k_0 a}H_n^{(2)}(\beta) \quad \kappa_2^2 H_n^{(2)}(\beta) \right]^T. \tag{46}$$

For ε-polarized outgoing waves

$$\mathbb{D}_{L,1}\mathbb{f}^\varepsilon_{\text{out},1} = \mathbb{h}^\varepsilon_{\text{out},1} \tag{47}$$

with

$$\mathbb{f}^\varepsilon_{\text{out},1} = [T^{12}_{\varepsilon\mu} \quad R^{121}_{\varepsilon\mu} \quad R^{121}_{\varepsilon\varepsilon} \quad T^{12}_{\varepsilon\varepsilon}]^T, \tag{48}$$

$$\mathbb{h}^\varepsilon_{\text{out},1} = \left[-\kappa_1 H_n^{(1)\prime}(\beta) \quad 0 \quad \frac{inh}{k_0 a}H_n^{(1)}(\beta) \quad -\kappa_1^2 H_n^{(1)}(\beta) \right]^T. \tag{49}$$

The Fresnel coefficients $T^{12}_{\varepsilon\mu}$, $R^{121}_{\varepsilon\mu}$, $R^{121}_{\varepsilon\varepsilon}$, $T^{12}_{\varepsilon\varepsilon}$, $R^{212}_{\varepsilon\varepsilon}$, $T^{21}_{\varepsilon\varepsilon}$, $T^{21}_{\varepsilon\mu}$, and $R^{212}_{\varepsilon\mu}$ are also listed in Appendix A.

C. Generalized Debye series expansion

This section is devoted to the derivation of GDSE that expresses the global scattering process in a series of local scattering processes, namely expresses the Mie scattering coefficients in a series of Fresnel coefficients.

In order to employ the above fictitious local scattering processes to expand Mie scattering coefficients in terms of Fresnel coefficients, it is necessary to isolate the internal outgoing waves $H_n^{(1)}(\beta)$ and local operator $\mathbb{D}_{L,1}$ from the operator $\mathbb{D}_1$. By using the relations (13), the operator $\mathbb{D}_1$ expressed in Eq. (22), which contains the incoming waves $H_n^{(2)}(\beta)$, outgoing waves $H_n^{(1)}(\alpha)$ and $H_n^{(1)}(\beta)$, can be separated as

$$\mathbb{D}_1 = \mathbb{D}_{L,1} - \mathbb{H}_{\text{int},1}, \tag{50}$$

where $\mathbb{D}_{L,1}$ is the local operator given by Eq. (34) and contains the incoming waves $H_n^{(2)}(\beta)$ and outgoing waves $H_n^{(1)}(\alpha)$ and $\mathbb{H}^\mu_{\text{int},1}$ contains only the outgoing waves $H_n^{(1)}(\beta)$. Because of the cross polarization, there are two kinds of outgoing waves $H_n^{(1)}(\beta)$ with different polarizations, so $\mathbb{H}_{\text{int},1}$ consists of two parts. The 4×4 matrix $\mathbb{H}_{\text{int},1}$ is composed of two operators $\mathbb{h}^\mu_{\text{out},1}$ and $\mathbb{h}^\varepsilon_{\text{out},1}$ defined by Eqs. (43) and (49) as

$$\mathbb{H}_{\text{int},1} = [\mathbb{0}, \mathbb{h}^\mu_{\text{out},1}, \mathbb{h}^\varepsilon_{\text{out},1}, \mathbb{0}]. \tag{51}$$

By inserting Eqs. (41) and (47) in Eq. (50), we can write

$$\mathbb{D}_1 = \mathbb{D}_{L,1}(\mathbb{I} - \mathbb{F}_{\text{int},1}), \tag{52}$$

$\mathbb{I}$ is the identity matrix and $\mathbb{F}_{\text{int},1}$ is defined by

$$\mathbb{F}_{\text{int},1} = [\mathbb{0}, \mathbb{f}^\mu_{\text{out},1}, \mathbb{f}^\varepsilon_{\text{out},1}, \mathbb{0}]. \tag{53}$$

By means of Eqs. (21), (33), and (52). the solution of the problem is finally expressed by

$$\mathbb{x}_1^\mu = (\mathbb{I} - \mathbb{F}_{\text{int},1})^{-1}\mathbb{f}^\mu_{\text{in},1} \tag{54}$$

which can then be developed in series form,

$$\mathbb{x}_1^{\mu} = \sum_{p=0}^{\infty} [\mathbb{F}_{\mathrm{int},1}]^p \mathbb{f}_{\mathrm{in},1}^{\mu}. \tag{55}$$

Equation (55) is the GDSE of problem concerning the μ-polarized incident waves. The term of mode p, as in DSE, corresponds to the wave which has penetrated in the scatterer and has been submitted to p local interactions before leaving the scatterer. By solving Eq. (54), the scattering coefficients can be expressed in Fresnel coefficients,

$$S_{n,1}^{\mu} = R_{\mu\mu}^{212} + \frac{1}{D_C}[T_{\mu\mu}^{12}(1 - R_{\varepsilon\varepsilon}^{121}) + R_{\mu\varepsilon}^{121} T_{\varepsilon\mu}^{12}] T_{\mu\mu}^{21} + \frac{1}{D_C}[T_{\varepsilon\mu}^{12}(1 - R_{\mu\mu}^{121}) + T_{\mu\mu}^{12} R_{\varepsilon\mu}^{121}] T_{\mu\varepsilon}^{21}, \tag{56}$$

$$C_{n,1}^{\mu} = \frac{1}{D_C}[(1 - R_{\varepsilon\varepsilon}^{121}) T_{\mu\mu}^{21} + R_{\varepsilon\mu}^{121} T_{\mu\varepsilon}^{21}], \tag{57}$$

$$D_{n,1}^{\mu} = \frac{1}{D_C}[(1 - R_{\mu\mu}^{121}) T_{\mu\varepsilon}^{21} + R_{\mu\varepsilon}^{121} T_{\mu\mu}^{21}], \tag{58}$$

$$Q_{n,1}^{\mu} = R_{\mu\varepsilon}^{212} + \frac{1}{D_C}[T_{\mu\varepsilon}^{12}(1 - R_{\varepsilon\varepsilon}^{121}) + T_{\varepsilon\varepsilon}^{12} R_{\mu\varepsilon}^{121})] T_{\mu\mu}^{21} + \frac{1}{D_C}[T_{\varepsilon\varepsilon}^{12}(1 - R_{\mu\mu}^{121}) + R_{\varepsilon\mu}^{121} T_{\mu\varepsilon}^{12})] T_{\mu\varepsilon}^{21}, \tag{59}$$

with

$$D_C = [1 - R_{\varepsilon\varepsilon}^{121}][1 - R_{\mu\mu}^{121}] - R_{\mu\varepsilon}^{121} R_{\varepsilon\mu}^{121}. \tag{60}$$

For the case of ε-polarized wave incidence, we can obtain the solution with the similar way written as

$$\mathbb{x}_1^{\varepsilon} = (\mathbb{I} - \mathbb{F}_{\mathrm{int},1})^{-1} \mathbb{f}_{\mathrm{in},1}^{\varepsilon} \tag{61}$$

or developed in series form

$$\mathbb{x}_1^{\varepsilon} = \sum_{p=0}^{\infty} [\mathbb{F}_{\mathrm{int},1}]^p \mathbb{f}_{\mathrm{in},1}^{\varepsilon}. \tag{62}$$

Here again, each term presents the contribution of a single mode.

By solving Eq. (61), we can obtain the final scattering coefficients

$$S_{n,1}^{\varepsilon} = R_{\varepsilon\varepsilon}^{212} + \frac{1}{D_C}[T_{\varepsilon\varepsilon}^{12}(1 - R_{\mu\mu}^{121}) + R_{\varepsilon\mu}^{121} T_{\mu\varepsilon}^{12}] T_{\varepsilon\varepsilon}^{21} + \frac{1}{D_C}[T_{\mu\varepsilon}^{12}(1 - R_{\varepsilon\varepsilon}^{121}) + T_{\varepsilon\varepsilon}^{12} R_{\mu\varepsilon}^{121}] T_{\varepsilon\mu}^{21}, \tag{63}$$

$$D_{n,1}^{\varepsilon} = \frac{1}{D_C}[(1 - R_{\varepsilon\varepsilon}^{121}) T_{\varepsilon\mu}^{21} + R_{\varepsilon\mu}^{121} T_{\varepsilon\varepsilon}^{21}], \tag{64}$$

$$C_{n,1}^{\varepsilon} = \frac{1}{D_C}[(1 - R_{\mu\mu}^{121}) T_{\varepsilon\varepsilon}^{21} + R_{\mu\varepsilon}^{121} T_{\varepsilon\mu}^{21}], \tag{65}$$

and

$$Q_{n,1}^{\varepsilon} = R_{\varepsilon\mu}^{212} + \frac{1}{D_C}[T_{\varepsilon\mu}^{12}(1 - R_{\mu\mu}^{121}) + T_{\mu\mu}^{12} R_{\varepsilon\mu}^{121}] T_{\varepsilon\varepsilon}^{21} + \frac{1}{D_C}[T_{\mu\mu}^{12}(1 - R_{\varepsilon\varepsilon}^{121}) + R_{\mu\varepsilon}^{121} T_{\varepsilon\mu}^{12}] T_{\mu\varepsilon}^{21}. \tag{66}$$

In the case of normal incidence, $h=0$, the results of Lock [20] are recovered.

III. COATED CYLINDER

Consider now a coated cylinder embedded in a medium (region 3) of refractive index m_3 illuminated by a plane wave which propagates in $x-z$ plane and makes an angle ξ with positive x axis. The radius and refractive index of the core (region 1) are, respectively, r_1 and m_1, and the radius and refractive index of the coating (region 2) are r_2 and m_2.

When the plane wave interacts with the cylinder, the total waves in each region are expressed as

$$\psi_3^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n [H_n^{(2)}(x_3) + S_{n,2}^{\mu} H_n^{(1)}(x_3)], \tag{67}$$

$$\psi_3^{\varepsilon} = -\sum_{n=-\infty}^{\infty} \mathcal{E}_n Q_{n,2}^{\mu} H_n^{(1)}(x_3), \tag{68}$$

$$\psi_2^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n [E_{n,2}^{\mu} H_n^{(2)}(x_2) - G_{n,2}^{\mu} H_n^{(1)}(x_2)], \tag{69}$$

$$\psi_2^{\varepsilon} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n [F_{n,2}^{\mu} H_n^{(2)}(x_2) - H_{n,2}^{\mu} H_n^{(1)}(x_2)], \tag{70}$$

$$\psi_1^{\mu} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n C_{n,2}^{\mu} J_n(x_1), \tag{71}$$

$$\psi_1^{\varepsilon} = \sum_{n=-\infty}^{\infty} \mathcal{E}_n D_{n,2}^{\mu} J_n(x_1), \tag{72}$$

where $E_{n,2}^{\mu}, G_{n,2}^{\mu}, F_{n,2}^{\mu}, H_{n,2}^{\mu}$ are the internal coefficients in the coating. The fields must be matched at each of the two boundaries $r=r_1$ and $r=r_2$. Applying the boundary conditions, we can obtain

$$\mathbb{D}_2 \mathbb{x}_2^{\mu} = \mathbb{h}_{\mathrm{ext},2}^{\mu}, \tag{73}$$

where

$$\mathbb{D}_2 = \begin{bmatrix} \mathbb{D}_{L,2}^{(1)} & \mathbb{H}_{\text{int},2}^{(1)} \\ \mathbb{H}_{\text{int},2}^{(2)} & \mathbb{D}_{L,2}^{(2)} \end{bmatrix}, \tag{74}$$

$$\mathbb{x}_2^{\mu} = \left[S_{n,2}^{\mu} \quad E_{n,2}^{\mu} \quad F_{n,2}^{\mu} \quad Q_{n,2}^{\mu} \quad G_{n,2}^{\mu} \quad \tfrac{1}{2}C_{n,2}^{\mu} \quad \tfrac{1}{2}D_{n,2}^{\mu} \quad H_{n,2}^{\mu} \right]^T, \tag{75}$$

$$\mathbb{h}_{\text{ext},2}^{\mu} = \left[\frac{inh}{k_3 r_2} H_n^{(2)}(\alpha_2) \quad \frac{\kappa_3^2}{m_3} H_n^{(2)}(\alpha_2) \quad m_3\kappa_3 H_n^{(2)\prime}(\alpha_2) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right]^T \tag{76}$$

with

$$\mathbb{D}_{L,2}^{(1)} = \begin{bmatrix} \frac{inh}{k_3 r_2} H_n^{(1)}(\alpha_2) & \frac{inh}{k_2 r_2} H_n^{(2)}(\beta_2) & \kappa_2 H_n^{(2)\prime}(\kappa_2 r_2) & \kappa_3 H_n^{(1)\prime}(\alpha_2) \\ \frac{\kappa_3^2}{m_3} H_n^{(1)}(\alpha_2) & \frac{\kappa_2^2}{m_2} H_n^{(2)}(\beta_2) & 0 & 0 \\ m_3\kappa_3 H_n^{(1)\prime}(\alpha_2) & m_2\kappa_2 H_n^{(2)\prime}(\beta_2) & -\frac{inh}{k_0 r_2} H_n^{(2)}(\beta_2) & -\frac{inh}{k_0 r_2} H_n^{(1)}(\alpha_2) \\ 0 & 0 & \kappa_2^2 H_n^{(2)}(\beta_2) & \kappa_3^2 H_n^{(1)}(\alpha_2) \end{bmatrix} \tag{77}$$

$$\mathbb{D}_{L,2}^{(2)} = \begin{bmatrix} \frac{inh}{k_2 r_1} H_n^{(1)}(\alpha_1) & 2\frac{inh}{k_1 r_1} J_n(\beta_1) & 2\kappa_1 J_n'(\beta_1) & \kappa_2 H_n^{(1)\prime}(\alpha_1) \\ \frac{\kappa_2^2}{m_2} H_n^{(1)}(\alpha_1) & 2\frac{\kappa_1^2}{m_1} J_n(\beta_1) & 0 & 0 \\ m_2\kappa_2 H_n^{(1)\prime}(\alpha_1) & 2m_1\kappa_1 J_n(\beta_1) & -2\frac{inh}{k_0 r_1} J_n(\beta_1) & -\frac{inh}{k_0 r_1} H_n^{(1)}(\alpha_1) \\ 0 & 0 & -2\kappa_1^2 J_n(\beta_1) & -\kappa_2^2 H_n^{(1)}(\alpha_1) \end{bmatrix}, \tag{78}$$

$$\mathbb{H}_{\text{int},2}^{(1)} = \begin{bmatrix} -\frac{inh}{k_2 r_2} H_n^{(1)}(\beta_2) & 0 & 0 & -\kappa_2 H_n^{(1)\prime}(\beta_2) \\ -\frac{\kappa_2^2}{m_2} H_n^{(1)}(\beta_2) & 0 & 0 & 0 \\ -m_2\kappa_2 H_n^{(1)\prime}(\beta_2) & 0 & 0 & \frac{inh}{k_0 r_2} H_n^{(1)}(\beta_2) \\ 0 & 0 & 0 & -\kappa_2^2 H_n^{(1)}(\beta_2) \end{bmatrix}, \tag{79}$$

$$\mathbb{H}_{\text{int},2}^{(2)} = \begin{bmatrix} 0 & -\frac{inh}{k_2 r_1} H_n^{(2)}(\alpha_1) & -\kappa_2 H_n^{(2)\prime}(\alpha_1) & 0 \\ 0 & -\frac{\kappa_2^2}{m_2} H_n^{(2)}(\alpha_1) & 0 & 0 \\ 0 & -m_2\kappa_2 H_n^{(2)\prime}(\alpha_1) & \frac{inh}{k_0 r_1} H_n^{(2)}(\alpha_1) & 0 \\ 0 & 0 & \kappa_2^2 H_n^{(2)}(\alpha_1) & 0 \end{bmatrix}, \tag{80}$$

$\mathbb{D}_2$ is the global operator introduced by the global boundary conditions and is separated into four parts. $\mathbb{D}_{L,2}^{(1)}$ is the operator of local interactions at the first interface (interface between regions 2 and 3) which contains the incoming waves $H_n^{(1)}(\alpha_2)$ and outgoing waves $H_n^{(2)}(\beta_2)$, while $\mathbb{D}_{L,2}^{(2)}$ is the operator of the global interaction at the second interface (interface between regions 1 and 2). $\mathbb{H}_{\text{int},2}^{(1)}$ is the operator of the internal excitations at the first interface which contains the outgoing waves $H_n^{(1)}(\beta_2)$. $\mathbb{H}_{\text{int},2}^{(2)}$ is the operator of the internal excitation at the second interface which contains incoming waves $H_n^{(2)}(\alpha_1)$. Then $\mathbb{D}_2$ can be separated as follows:

$$\mathbb{D}_2 = \mathbb{D}_{L,2} + \mathbb{H}_{\mathrm{int},2}, \tag{81}$$

where

$$\mathbb{D}_{L,2} = \begin{bmatrix} \mathbb{D}_{L,2}^{(1)} & \mathbb{0} \\ \mathbb{0} & \mathbb{D}_{L,2}^{(2)} \end{bmatrix}, \tag{82}$$

$$\mathbb{H}_{\mathrm{int},2} = \begin{bmatrix} \mathbb{0} & \mathbb{H}_{\mathrm{int},2}^{(2)} \\ \mathbb{H}_{\mathrm{int},2}^{(1)} & \mathbb{0} \end{bmatrix}. \tag{83}$$

The local interaction between the external excitation and the first interface is contained in $\mathbb{h}_{\mathrm{in},2}^{\mu}$ which has similar definition as $\mathbb{h}_{\mathrm{in},1}^{\mu}$, and given by the equation system

$$\mathbb{D}_{L,2}\mathbb{f}_{\mathrm{in},2}^{\mu} = \mathbb{h}_{\mathrm{in},2}^{\mu}. \tag{84}$$

$\mathbb{f}_{\mathrm{in},2}^{\mu}$ is the vector of "primary interaction" and contains the response of the first interface to the external excitation expressed by the external reflection and transmission coefficients. It can be obtained from Eq. (33),

$$\mathbb{f}_{\mathrm{in},2}^{\mu} = [R_{\mu\mu}^{323} \quad T_{\mu\mu}^{32} \quad T_{\mu\varepsilon}^{32} \quad R_{\mu\varepsilon}^{323} \quad 0 \quad 0 \quad 0 \quad 0]^T, \tag{85}$$

where the Fresnel coefficients $R_{\mu\mu}^{323}$, $R_{\mu\varepsilon}^{323}$, $T_{\mu\mu}^{32}$, and $T_{\mu\varepsilon}^{32}$ are similar to those for a homogeneous cylinder given in Appendix A.

The equation system for excitations in region 2 is

$$\mathbb{D}_{L,2}^{\mu}\mathbb{F}_2^{\mu} = -\mathbb{H}_{\mathrm{int},2}^{\mu}, \tag{86}$$

where $\mathbb{F}_2^{\mu}$ is an operator containing all local internal reflection and transmission coefficients at the first interface and all coefficients describing the global scattering process of the core to the excitations in region 2. So the excitations in region 2 can be divided into two parts: One is the internal excitations to the first interface, and the other is the global excitations to the core. Therefore, the operator $\mathbb{F}_2^{\mu}$ can be obtained from two fictitious scattering processes: Internal excitation to first interface, excitations imposed to the core.

The equation system for scattering of internal excitations is

$$\mathbb{D}_{L,2}^{\mu}\mathbb{F}_2^{\mu(1)} = -\mathbb{H}_{\mathrm{int},2}^{\mu(1)}, \tag{87}$$

where

$$\mathbb{H}_{\mathrm{int},2}^{\mu(1)} = [\mathbb{h}_{\mathrm{out},2}^{\mu} \quad 0 \quad 0 \quad \mathbb{h}_{\mathrm{out},2}^{\varepsilon}]. \tag{88}$$

$\mathbb{F}_2^{\mu(1)}$ contains all the local internal reflection and transmission coefficients at the first interface, and can be obtained by replacing Eqs. (41) and (47) in Eq. (87),

$$\mathbb{F}_2^{\mu(1)} = [\mathbb{f}_{\mathrm{out},2}^{\mu} \quad 0 \quad 0 \quad \mathbb{f}_{\mathrm{out},2}^{\varepsilon}], \tag{89}$$

$\mathbb{h}_{\mathrm{out},2}^{\mu}$, $\mathbb{h}_{\mathrm{out},2}^{\varepsilon}$, $\mathbb{f}_{\mathrm{out},2}^{\mu}$, and $\mathbb{f}_{\mathrm{out},2}^{\varepsilon}$ have similar definition with $\mathbb{h}_{\mathrm{out},1}^{\mu}$, $\mathbb{h}_{\mathrm{out},1}^{\varepsilon}$, $\mathbb{f}_{\mathrm{out},1}^{\mu}$, and $\mathbb{f}_{\mathrm{out},1}^{\varepsilon}$.

The excitations imposed to the core is written as

$$\mathbb{D}_{L,2}^{\mu}\mathbb{F}_2^{\mu(2)} = -\mathbb{H}_{\mathrm{int},2}^{\mu(2)}, \tag{90}$$

where

$$\mathbb{H}_{\mathrm{int},2}^{\mu(2)} = [\mathbb{h}_{\mathrm{ext},1}^{\mu} \quad 0 \quad 0 \quad \mathbb{h}_{\mathrm{ext},1}^{\varepsilon}]. \tag{91}$$

$\mathbb{F}_2^{\mu(2)}$ contains the response of the core to the excitations, and can be obtained by replacing Eqs. (21) and (26) into Eq. (90),

$$\mathbb{F}_2^{\mu(2)} = [0 \quad \mathbb{x}_1^{\mu} \quad \mathbb{x}_1^{\varepsilon} \quad 0]. \tag{92}$$

The comparison of Eq. (92) with Eqs. (23) and (27) shows that $\mathbb{F}_2^{\mu(2)}$ gives the effect of the core to whole coated cylinder scattering [18,19]. Then, the operator $\mathbb{F}_2^{\mu}$ is finally obtained from Eqs. (89) and (92),

$$\mathbb{F}_2^{\mu} = \begin{bmatrix} 0 & \mathbb{F}_2^{\mu(1)} \\ \mathbb{F}_2^{\mu(2)} & 0 \end{bmatrix}. \tag{93}$$

Employing the similar derivation, we can obtain the final GDSE for coated cylinders as follows:

$$\mathbb{x}_2^{\mu} = [\mathbb{I} - \mathbb{F}_2^{\mu}]^{-1}\mathbb{f}_{\mathrm{in},2}^{\mu} \tag{94}$$

or

$$\mathbb{x}_2^{\mu} = \sum_{p=0}^{\infty} [\mathbb{F}_2^{\mu}]^p \mathbb{f}_{\mathrm{in},2}^{\mu}. \tag{95}$$

For the case of ε-polarized wave incidence, the GDSE is expressed as

$$\mathbb{x}_2^{\varepsilon} = [\mathbb{I} - \mathbb{F}_2^{\varepsilon}]^{-1}\mathbb{f}_{\mathrm{in},2}^{\varepsilon} \tag{96}$$

or

$$\mathbb{x}_2^{\varepsilon} = \sum_{p=0}^{\infty} [\mathbb{F}_2^{\varepsilon}]^p \mathbb{f}_{\mathrm{in},2}^{\varepsilon} \tag{97}$$

with

$$\mathbb{x}_2^{\varepsilon} = \left[Q_{n,2}^{\varepsilon} \quad F_{n,2}^{\varepsilon} \quad E_{n,2}^{\varepsilon} \quad S_{n,2}^{\varepsilon} \quad H_{n,2}^{\varepsilon} \quad \tfrac{1}{2}D_{n,2}^{\varepsilon} \quad \tfrac{1}{2}C_{n,2}^{\varepsilon} \quad G_{n,2}^{\varepsilon}\right]^T, \tag{98}$$

$$\mathbb{f}_{\mathrm{in},2}^{\varepsilon} = [R_{\varepsilon\mu}^{323} \quad T_{\varepsilon\mu}^{32} \quad T_{\varepsilon\varepsilon}^{32} \quad R_{\varepsilon\varepsilon}^{323} \quad 0 \quad 0 \quad 0 \quad 0]^T, \tag{99}$$

$$\mathbb{F}_2^{\varepsilon} = \mathbb{F}_2^{\mu}. \tag{100}$$

Equations (94)–(97) give the GDSE for coated cylinder scattering, on the basis of which the GDSE for multilayered cylinder scattering is derived in the next section.

IV. MULTILAYERED CYLINDER

An l-layered cylinder can be considered as a coated cylinder whose coating is the layer l and whose core is the $l-1$ layered cylinder [18,19], so we can obtain the GDSE for multilayered cylinder scattering on the basis of GDSE for a coated one. For μ-polarized wave incidence, the GDSE is expressed as

$$\mathbb{x}_l^{\mu} = [\mathbb{I} - \mathbb{F}_l^{\mu}]^{-1}\mathbb{f}_{\mathrm{in},l}^{\mu} \tag{101}$$

or developed in series form

$$\mathbb{x}_l^{\mu} = \sum_{p=0}^{\infty} [\mathbb{F}_l^{\mu}]^p \mathbb{f}_{\text{in},l}^{\mu}, \tag{102}$$

where

$$\mathbb{x}_l^{\mu} = \begin{bmatrix} S_{n,l}^{\mu} & E_{n,l}^{\mu} & F_{n,l}^{\mu} & Q_{n,l}^{\mu} & G_{n,l}^{\mu} & \frac{1}{2}C_{n,l}^{\mu} & \frac{1}{2}D_{n,l}^{\mu} & H_{n,l}^{\mu} \end{bmatrix}^T, \tag{103}$$

$$\mathbb{F}_l^{\mu} = \begin{bmatrix} 0 & \mathbb{F}_l^{\mu(1)} \\ \mathbb{F}_l^{\mu(2)} & 0 \end{bmatrix}, \tag{104}$$

$$\mathbb{F}_l^{\mu(1)} = \begin{bmatrix} T_{\mu\mu}^{l,l+1} & 0 & 0 & T_{\varepsilon\mu}^{l,l+1} \\ R_{\mu\mu}^{l,l+1,l} & 0 & 0 & R_{\varepsilon\mu}^{l,l+1,l} \\ R_{\mu\varepsilon}^{l,l+1,l} & 0 & 0 & R_{\varepsilon\varepsilon}^{l,l+1,l} \\ T_{\mu\varepsilon}^{l,l+1} & 0 & 0 & T_{\varepsilon\varepsilon}^{l,l+1} \end{bmatrix}, \tag{105}$$

$$\mathbb{F}_l^{\mu(2)} = [0 \quad \mathbb{x}_{l-1}^{\mu} \quad \mathbb{x}_{l-1}^{\varepsilon} \quad 0], \tag{106}$$

$$\mathbb{f}_{\text{in},l}^{\mu} = [R_{\mu\mu}^{l+1,l,l+1} \quad T_{\mu\mu}^{l+1,l} \quad T_{\mu\varepsilon}^{l+1,l} \quad R_{\mu\varepsilon}^{l+1,l,l+1} \quad 0 \quad 0 \quad 0 \quad 0]^T. \tag{107}$$

The superscript l represents that the cylinder is l layered. $\mathbb{F}_l^{\mu(1)}$ is an operator containing all the local internal reflection and transmission coefficients at the interface between layers l and $l+1$ (surrounding). $\mathbb{F}_l^{\mu(2)}$ contains the response of the core to the excitations imposed to the core. $\mathbb{f}_{\text{in},l}^{\mu}$ contains the response of the interface between layer l and $l+1$ to the external excitations. The Fresnel coefficients are similar to those for a homogeneous cylinder scattering. All scattering coefficients can be expressed as the Fresnel coefficients according to Eq. (101),

$$\begin{aligned} S_{n,l}^{\mu} = R_{\mu\mu}^{l+1,l,l+1} + \frac{1}{D_s}\{[(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})(R_{\mu\varepsilon}^{l,l+1,l}T_{\varepsilon\mu}^{l,l+1} \\ - T_{\mu\mu}^{l,l+1}R_{\varepsilon\varepsilon}^{l,l+1,l}) - T_{\mu\mu}^{l,l+1}S_{n,l-1}^{\mu} - Q_{n,l-1}^{\mu}T_{\varepsilon\mu}^{l,l+1}]T_{\mu\mu}^{l+1,l} \\ + [(S_{n,l-1}^{\varepsilon}S_{n,l-1}^{\mu} - Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})(T_{\mu\mu}^{l,l+1}R_{\varepsilon\mu}^{l,l+1,l} - R_{\mu\mu}^{l,l+1,l}T_{\varepsilon\mu}^{l,l+1}) \\ - Q_{n,l-1}^{\varepsilon}T_{\mu\mu}^{l,l+1} - T_{\varepsilon\mu}^{l,l+1}S_{n,l-1}^{\varepsilon}]T_{\mu\varepsilon}^{l+1,l}\}, \end{aligned} \tag{108}$$

$$\begin{aligned} Q_{n,l}^{\mu} = R_{\mu\varepsilon}^{l+1,l,l+1} + \frac{1}{D_s}\{[(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})(R_{\mu\varepsilon}^{l,l+1,l}T_{\varepsilon\varepsilon}^{l,l+1} \\ - T_{\mu\varepsilon}^{l,l+1}R_{\varepsilon\varepsilon}^{l,l+1,l}) - Q_{n,l-1}^{\mu}T_{\varepsilon\varepsilon}^{l,l+1} - T_{\mu\varepsilon}^{l,l+1}S_{n,l-1}^{\mu}]T_{\mu\mu}^{l+1,l} \\ + [(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})(T_{\mu\varepsilon}^{l,l+1}R_{\varepsilon\mu}^{l,l+1,l} - T_{\varepsilon\varepsilon}^{l,l+1}R_{\mu\mu}^{l,l+1,l}) \\ - T_{\mu\varepsilon}^{l,l+1}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}T_{\varepsilon\varepsilon}^{l,l+1}]T_{\mu\varepsilon}^{l+1,l}\}, \end{aligned} \tag{109}$$

$$\begin{aligned} E_{n,l}^{\mu} = \frac{1}{D_s}[(-1 + R_{\mu\varepsilon}^{l,l+1,l}Q_{n,l-1}^{\varepsilon} + R_{\varepsilon\varepsilon}^{l,l+1,l}S_{n,l-1}^{\varepsilon})T_{\mu\mu}^{l+1,l} \\ + (-S_{n,l-1}^{\varepsilon}R_{\varepsilon\mu}^{l,l+1,l} - R_{\mu\mu}^{l,l+1,l}Q_{n,l-1}^{\varepsilon})T_{\mu\varepsilon}^{l+1,l}], \end{aligned} \tag{110}$$

$$\begin{aligned} F_{n,l}^{\mu} = \frac{1}{D_s}[(-R_{\mu\varepsilon}^{l,l+1,l}S_{n,l-1}^{\mu} - R_{\varepsilon\varepsilon}^{l,l+1,l}Q_{n,l-1}^{\mu})T_{\mu\mu}^{l+1,l} \\ + (R_{\mu\mu}^{l,l+1,l}S_{n,l-1}^{\mu} + R_{\varepsilon\mu}^{l,l+1,l}Q_{n,l-1}^{\mu} - 1)T_{\mu\varepsilon}^{l+1,l}], \end{aligned} \tag{111}$$

$$\begin{aligned} G_{n,l}^{\mu} = \frac{1}{D_s}\{[(S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\varepsilon\varepsilon}^{l,l+1,l} - S_{n,l-1}^{\mu}]T_{\mu\mu}^{l+1,l} \\ + [(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}S_{n,l-1}^{\mu})R_{\varepsilon\mu}^{l,l+1,l} - Q_{n,l-1}^{\varepsilon}]T_{\mu\varepsilon}^{l+1,l}\}, \end{aligned} \tag{112}$$

$$\begin{aligned} C_{n,l}^{\mu} = \frac{1}{D_s}\{[(C_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}D_{n,l-1}^{\varepsilon})R_{\mu\varepsilon}^{l,l+1,l} + (C_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} \\ - Q_{n,l-1}^{\mu}D_{n,l-1}^{\varepsilon})R_{\varepsilon\varepsilon}^{l,l+1,l} - C_{n,l-1}^{\mu}]T_{\mu\mu}^{l+1,l} + [(Q_{n,l-1}^{\mu}D_{n,l-1}^{\varepsilon} \\ - S_{n,l-1}^{\varepsilon}C_{n,l-1}^{\mu})R_{\varepsilon\mu}^{l,l+1,l} + (S_{n,l-1}^{\mu}D_{n,l-1}^{\varepsilon} - C_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\mu\mu}^{l,l+1,l} \\ - D_{n,l-1}^{\varepsilon}]T_{\mu\varepsilon}^{l+1,l}\}, \end{aligned} \tag{113}$$

$$\begin{aligned} D_{n,l}^{\mu} = \frac{1}{D_s}\{[(F_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon})R_{\mu\varepsilon}^{l,l+1,l} \\ + (F_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon})R_{\varepsilon\varepsilon}^{l,l+1,l} - F_{n,l-1}^{\mu}]T_{\mu\mu}^{l+1,l} \\ + [(Q_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}F_{n,l-1}^{\mu})R_{\varepsilon\mu}^{l,l+1,l} \\ + (S_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon} - F_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\mu\mu}^{l,l+1,l} - E_{n,l-1}^{\varepsilon}]T_{\mu\varepsilon}^{l+1,l}\}, \end{aligned} \tag{114}$$

$$\begin{aligned} H_{n,l}^{\mu} = \frac{1}{D_s}\{[(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})R_{\mu\varepsilon}^{l,l+1,l} - Q_{n,l-1}^{\mu}]T_{\mu\mu}^{l+1,l} \\ + [(S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\mu\mu}^{l,l+1,l} - S_{n,l-1}^{\varepsilon}]T_{\mu\varepsilon}^{l+1,l}\}, \end{aligned} \tag{115}$$

$$\begin{aligned} D_s = [1 - (S_{n,l-1}^{\varepsilon}R_{\varepsilon\varepsilon}^{l,l+1,l} - 1)(R_{\mu\mu}^{l,l+1,l}S_{n,l-1}^{\mu} - 1) - (Q_{n,l-1}^{\varepsilon}R_{\mu\varepsilon}^{l,l+1,l} \\ - 1)(Q_{n,l-1}^{\mu}R_{\varepsilon\mu}^{l,l+1,l} - 1) + R_{\mu\mu}^{l,l+1,l}Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon}R_{\varepsilon\varepsilon}^{l,l+1,l} \\ + S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon}R_{\mu\varepsilon}^{l,l+1,l}R_{\varepsilon\mu}^{l,l+1,l}] \end{aligned} \tag{116}$$

and they can be calculated by the upward recurrence.

For the ε-polarized wave incidence, the GDSE is written as

$$\mathbb{x}_l^{\varepsilon} = [\mathbb{I} - \mathbb{F}_l^{\varepsilon}]^{-1}\mathbb{f}_{\text{in},l}^{\varepsilon}, \tag{117}$$

or

$$\mathbb{x}_l^{\varepsilon} = \sum_{p=0}^{\infty} [\mathbb{F}_l^{\varepsilon}]^p \mathbb{f}_{\text{in},l}^{\varepsilon}, \tag{118}$$

where

$$\mathbb{x}_l^{\varepsilon} = \begin{bmatrix} Q_{n,l}^{\varepsilon} & F_{n,l}^{\varepsilon} & E_{n,l}^{\varepsilon} & S_{n,l}^{\varepsilon} & H_{n,l}^{\varepsilon} & \frac{1}{2}D_{n,l}^{\varepsilon} & \frac{1}{2}C_{n,l}^{\varepsilon} & G_{n,l}^{\varepsilon} \end{bmatrix}^T, \tag{119}$$

$$\mathbb{F}_l^{\varepsilon} = \begin{bmatrix} 0 & \mathbb{F}_l^{\varepsilon(1)} \\ \mathbb{F}_l^{\varepsilon(2)} & 0 \end{bmatrix}, \tag{120}$$

$$\mathbb{F}_l^{\varepsilon(1)} = \begin{bmatrix} T_{\mu\mu}^{l,l+1} & 0 & 0 & T_{\varepsilon\mu}^{l,l+1} \\ R_{\mu\mu}^{l,l+1,l} & 0 & 0 & R_{\varepsilon\mu}^{l,l+1,l} \\ R_{\mu\varepsilon}^{l,l+1,l} & 0 & 0 & R_{\varepsilon\varepsilon}^{l,l+1,l} \\ T_{\mu\varepsilon}^{l,l+1} & 0 & 0 & T_{\varepsilon\varepsilon}^{l,l+1} \end{bmatrix}, \tag{121}$$

$$\mathbb{F}_l^{\varepsilon(2)} = \mathbb{F}_l^{\mu(2)}, \tag{122}$$

$$\mathrm{f}_{\mathrm{in},l}^{\varepsilon} = [R_{\varepsilon\mu}^{l+1,l,l+1} \quad T_{\varepsilon\mu}^{l+1,l} \quad T_{\varepsilon\varepsilon}^{l+1,l} \quad R_{\varepsilon\varepsilon}^{l+1,l,l+1} \quad 0 \quad 0 \quad 0 \quad 0]^T. \tag{123}$$

The notations $\mathbb{F}_l^{\varepsilon(1)}$, $\mathbb{F}_l^{\varepsilon(2)}$, and $\mathrm{f}_{\mathrm{in},l}^{\varepsilon}$ have the similar meaning with that used for the μ-polarized wave incidence. All scattering coefficients are expressed in Fresnel coefficients according to Eq. (117),

$$S_{n,l}^{\varepsilon} = R_{\varepsilon\varepsilon}^{l+1,l,l+1} + \frac{1}{D_s}\{[(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})(R_{\mu\varepsilon}^{l,l+1,l}T_{\varepsilon\varepsilon}^{l,l+1} - T_{\mu\varepsilon}^{l,l+1}R_{\varepsilon\varepsilon}^{l,l+1,l}) - Q_{n,l-1}^{\mu}T_{\varepsilon\varepsilon}^{l,l+1} - T_{\mu\varepsilon}^{l,l+1}S_{n,l-1}^{\mu}]T_{\varepsilon\mu}^{l+1,l} + [(S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})(T_{\varepsilon\varepsilon}^{l,l+1}R_{\mu\mu}^{l,l+1,l} - R_{\varepsilon\mu}^{l,l+1,l}T_{\mu\varepsilon}^{l,l+1}) - T_{\varepsilon\varepsilon}^{l,l+1}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\varepsilon}T_{\mu\varepsilon}^{l,l+1}]T_{\varepsilon\varepsilon}^{l+1,l}\}, \tag{124}$$

$$Q_{n,l}^{\varepsilon} = R_{\varepsilon\mu}^{l+1,l,l+1} + \frac{1}{D_s}\{[(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})(R_{\mu\varepsilon}^{l,l+1,l}T_{\varepsilon\mu}^{l,l+1} - T_{\mu\mu}^{l,l+1}R_{\varepsilon\varepsilon}^{l,l+1,l}) - Q_{n,l-1}^{\mu}T_{\varepsilon\mu}^{l,l+1} - T_{\mu\mu}^{l,l+1}S_{n,l-1}^{\mu}]T_{\varepsilon\mu}^{l+1,l} + [(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}S_{n,l-1}^{\mu})(R_{\varepsilon\mu}^{l,l+1,l}T_{\mu\mu}^{l,l+1} - T_{\varepsilon\mu}^{l,l+1}R_{\mu\mu}^{l,l+1,l}) - Q_{n,l-1}^{\varepsilon}T_{\mu\mu}^{l,l+1} - T_{\varepsilon\mu}^{l,l+1}S_{n,l-1}^{\varepsilon}]T_{\varepsilon\varepsilon}^{l+1,l}\}, \tag{125}$$

$$F_{n,l}^{\varepsilon} = \frac{1}{D_s}[(-1 + R_{\mu\varepsilon}^{l,l+1,l}Q_{n,l-1}^{\varepsilon} + R_{\varepsilon\varepsilon}^{l,l+1,l}S_{n,l-1}^{\varepsilon})T_{\varepsilon\mu}^{l+1,l} + (-R_{\mu\mu}^{l,l+1,l}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}R_{\varepsilon\mu}^{l,l+1,l})T_{\varepsilon\varepsilon}^{l+1,l}], \tag{126}$$

$$E_{n,l}^{\varepsilon} = \frac{1}{D_s}[(-R_{\varepsilon\varepsilon}^{l,l+1,l}Q_{n,l-1}^{\mu} - R_{\mu\varepsilon}^{l,l+1,l}S_{n,l-1}^{\mu})T_{\varepsilon\mu}^{l+1,l} + (R_{\mu\mu}^{l,l+1,l}S_{n,l-1}^{\mu} + R_{\varepsilon\mu}^{l,l+1,l}Q_{n,l-1}^{\mu} - 1)T_{\varepsilon\varepsilon}^{l+1,l}], \tag{127}$$

$$H_{n,l}^{\varepsilon} = \frac{1}{D_s}\{[(S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\varepsilon\varepsilon}^{l,l+1,l} - S_{n,l-1}^{\mu}]T_{\varepsilon\mu}^{l+1,l} + [(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}S_{n,l-1}^{\mu})R_{\varepsilon\mu}^{l,l+1,l} - Q_{n,l-1}^{\varepsilon}]T_{\varepsilon\varepsilon}^{l+1,l}\}, \tag{128}$$

$$D_{n,l}^{\varepsilon} = \frac{1}{D_s}\{[(E_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}F_{n,l-1}^{\varepsilon})R_{\varepsilon\varepsilon}^{l,l+1,l} + (E_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}F_{n,l-1}^{\varepsilon})R_{\mu\varepsilon}^{l,l+1,l} - E_{n,l-1}^{\mu}]T_{\varepsilon\mu}^{l+1,l} + [(S_{n,l-1}^{\mu}F_{n,l-1}^{\varepsilon} - E_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\mu\mu}^{l,l+1,l} + (Q_{n,l-1}^{\mu}F_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}E_{n,l-1}^{\mu})R_{\varepsilon\mu}^{l,l+1,l} - F_{n,l-1}^{\varepsilon}]T_{\varepsilon\varepsilon}^{l+1,l}\}, \tag{129}$$

$$C_{n,l}^{\varepsilon} = \frac{1}{D_s}\{[(F_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon})R_{\varepsilon\varepsilon}^{l,l+1,l} + (F_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon})R_{\mu\varepsilon}^{l,l+1,l} - F_{n,l-1}^{\mu}]T_{\varepsilon\mu}^{l+1,l} + [(S_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon} - F_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\mu\mu}^{l,l+1,l} + (Q_{n,l-1}^{\mu}E_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\varepsilon}F_{n,l-1}^{\mu})R_{\varepsilon\mu}^{l,l+1,l} - E_{n,l-1}^{\varepsilon}]T_{\varepsilon\varepsilon}^{l+1,l}\}, \tag{130}$$

$$E_{n,l}^{\varepsilon} = \frac{1}{D_s}\{[(Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon} - S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon})R_{\mu\varepsilon}^{l,l+1,l} - Q_{n,l-1}^{\mu}]T_{\varepsilon\mu}^{l+1,l} + [(S_{n,l-1}^{\mu}S_{n,l-1}^{\varepsilon} - Q_{n,l-1}^{\mu}Q_{n,l-1}^{\varepsilon})R_{\mu\mu}^{l,l+1,l} - S_{n,l-1}^{\varepsilon}]T_{\varepsilon\varepsilon}^{l+1,l}\}. \tag{131}$$

It is noticeable that when $\xi=0$ the cylinders are normally illuminated by plane waves. By the relations

$$h = k_0 \sin \xi = 0, \tag{132}$$

and

$$\kappa_j = \sqrt{k_j^2 - h^2} = k_j, \tag{133}$$

all scattering coefficients for the cross polarization $Q_{n,l}^{\mu}$ and $Q_{n,l}^{\varepsilon}$ are 0, and all other scattering coefficients and Fresnel coefficients are the same as those for normal incidence [20], as expected.

V. NUMERICAL ALGORITHM

In principle, the scattering problem has been solved in the above sections and we can calculate all the fields with these formulas. But in numerical simulation, two pitfalls will be met in the calculation of Fresnel coefficients. First, when the imaginary part of x is very large. $H_n^{(1)}(x)$ will overflow. Second, when $n>|x|$, the upward recurrence algorithm will be unstable [2,29]. In order to overcome such pitfalls, an improved algorithm is presented.

First, we define the logarithmic derivatives of Hankel functions as [2,19,20]

$$D_n^{(1)}(x) = \frac{H_n^{(1)'}(x)}{H_n^{(1)}(x)}, \tag{134}$$

$$D_n^{(2)}(x) = \frac{H_n^{(2)'}(x)}{H_n^{(2)}(x)}. \tag{135}$$

The logarithmic derivatives can be well calculated by the following recurrence algorithm:

$$A_n(x) = 1/[(n-1)/x - A_{n-1}(x)] - n/x, \tag{136}$$

where $A_n(x)$ stands for $D_n^{(1)}(x)$ or $D_n^{(2)}(x)$. The initial values of $D_n^{(1)}(x)$ or $D_n^{(2)}(x)$ are, respectively,

$$D_0^{(1)}(x) = H_0^{(1)}(x)/H_1^{(1)}(x) - 1/x$$

and

$$D_0^{(2)}(x) = H_0^{(2)}(x)/H_1^{(2)}(x) - 1/x.$$

All Fresnel coefficients are expressed in terms of logarithmic derivatives, and listed in Appendix B.

From Appendix A, we find that all transmission coefficients $T_{ij}^{21}(i,j=\mu,\varepsilon)$ have the ratio of Hankel functions of second kinds $H_n^{(2)}(x_2)/H_n^{(2)}(x_1)$, and all T_{ij}^{12} have the ratio of Hankel functions of first kinds $H_n^{(1)}(x_1)/H_n^{(1)}(x_2)$. When n is larger than the parameter of Hankel functions, both the ratios $H_n^{(2)}(x_2)/H_n^{(2)}(x_1)$ and $H_n^{(1)}(x_1)/H_n^{(1)}(x_2)$ diverge quickly and cannot be calculated directly. By consideration of the physical scattering process and of Eqs. (108)–(116) and (124)–(131), we find that transmission coefficients T_{ij}^{21} and T_{ij}^{12} exist together in all scattering coefficients. Therefore, we define T_n as the product of $H_n^{(2)}(x_2)/H_n^{(2)}(x_1)$ and $H_n^{(1)}(x_1)/H_n^{(1)}(x_2)$,

$$T_n = \frac{H_n^{(2)}(x_2)}{H_n^{(1)}(x_2)}\frac{H_n^{(1)}(x_1)}{H_n^{(2)}(x_1)}. \tag{137}$$

The ratios $H_n^{(2)}(x_2)/H_n^{(2)}(x_1)$ and $H_n^{(1)}(x_1)/H_n^{(1)}(x_2)$ can then be obtained by the ratio $H_n^{(1)}(z)/H_n^{(2)}(z)$ ($z=x_2$ or x_1), which can be calculated by the following recurrence relation:

$$\frac{H_n^{(1)}(z)}{H_n^{(2)}(z)} = \frac{H_{n-1}^{(1)}(z)}{H_{n-1}^{(2)}(z)}\frac{D_n^{(2)}(z)+n/z}{D_n^{(1)}(z)+n/z} \tag{138}$$

with initial value $H_0^{(1)}(z)/H_0^{(2)}(z)$. The numerical evaluation by this equation is stable.

VI. FAR-FIELD SCATTERED INTENSITY

When all scattering coefficients are obtained, we can calculate the far-field scattered intensity according to the following formulas:

$$I_{\mu\mu}^l(\theta) = |S_{\mu\mu}^l(\theta)|^2, \tag{139}$$

$$I_{\varepsilon\varepsilon}^l(\theta) = |S_{\varepsilon\varepsilon}^l(\theta)|^2, \tag{140}$$

$$I_{\mu\varepsilon}^l(\theta) = |S_{\mu\varepsilon}^l(\theta)|^2, \tag{141}$$

$$I_{\varepsilon\mu}^l(\theta) = |S_{\varepsilon\mu}^l(\theta)|^2, \tag{142}$$

where $I_{\mu\mu}^l$, $I_{\varepsilon\varepsilon}^l$, $I_{\mu\varepsilon}^l$, and $I_{\varepsilon\mu}^l$ are, respectively, the scattered intensities corresponding to the μ-polarization-preserving scattering, ε-polarization-preserving scattering, and cross-polarization scatterings. S^l's are the corresponding scattering amplitudes written as

$$S_{\mu\mu}^l = B_0^\mu + 2\sum_{n=1}^{\infty} B_{n,l}^\mu \cos(n\theta), \tag{143}$$

$$S_{\varepsilon\varepsilon}^l = B_0^\varepsilon + 2\sum_{n=1}^{\infty} B_{n,l}^\varepsilon \cos(n\theta), \tag{144}$$

$$S_{\mu\varepsilon}^l = 2i\sum_{n=1}^{\infty} Q_{n,l}^\mu \sin(n\theta), \tag{145}$$

$$S_{\varepsilon\mu}^l = 2i\sum_{n=1}^{\infty} Q_{n,l}^\varepsilon \sin(n\theta). \tag{146}$$

Because the scattering amplitudes for the two cross-polarization channels differ only in sign, namely $S_{\mu\varepsilon}^l=-S_{\varepsilon\mu}^l$ [30], the scattered intensities for two channels are the same ($I_{\mu\varepsilon}^l=I_{\varepsilon\mu}^l$).

To verify our algorithm and code, the GDSE is first employed to the study of the simplest case, namely homogeneous cylinders obliquely illuminated by plane waves. Liou [31] calculated the far-field scattered intensities. We compare our results of GDSE with theirs. Figure 4 depicts the far-field scattered intensities for absorbing homogeneous ice cylinders obliquely illuminated by plane waves with different tilt angles that correspond to Figs. 4 and 6 of Ref. [31]. The parameters used are $a=10$ μm, $\lambda=10$ μm, $m=1.152-0.0413i$, and $\xi=5°$, 45°, and 85°, respectively. We find that our results are in good agreement with that of Liou. It is worth to note that the cross-polarized intensity [Fig. 4(c)] is very important in oblique-cylinder scattering and is absent in

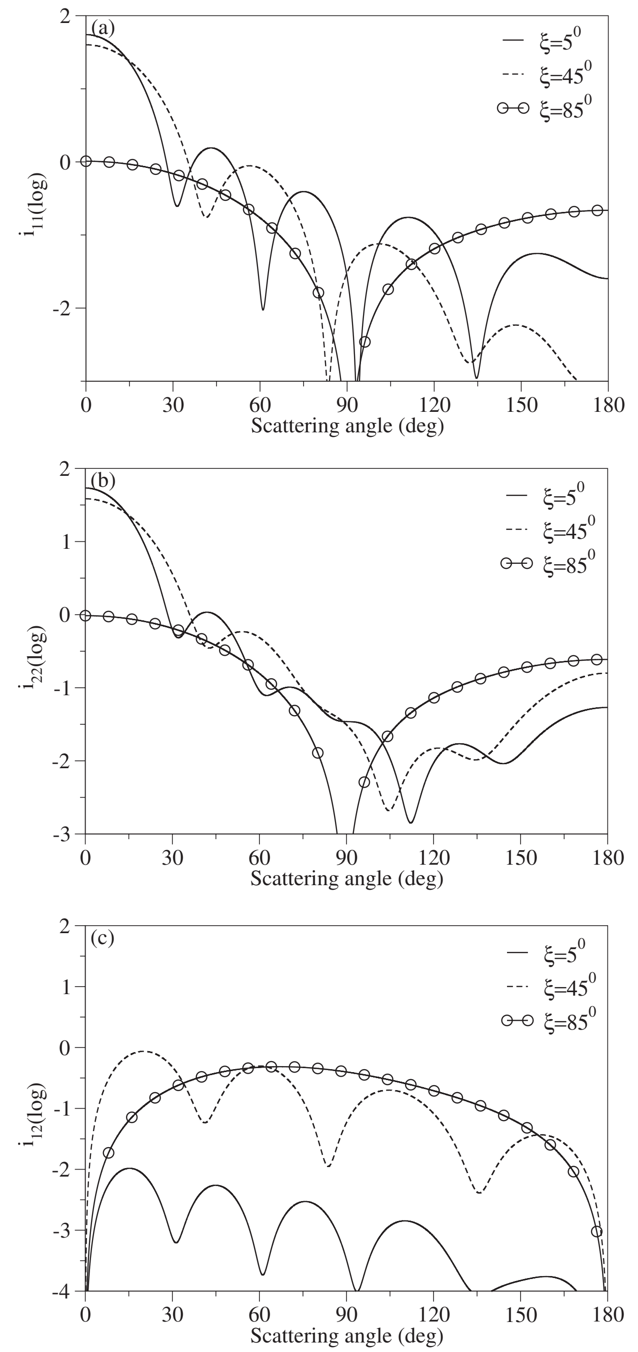


FIG. 4. Far-field scattered intensity for absorbing homogeneous ice cylinders obliquely illuminated by plane waves with different tilted angles ($a=10$ μm, $\lambda=10$ μm, $m=1.152-0.0413i$). The log scale is to the base 10.

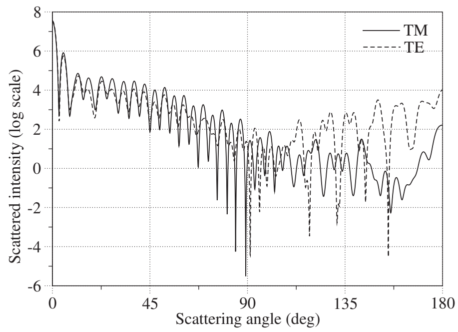


FIG. 5. Far-field scattered intensity for a doubly clad image-transmitting fiber obliquely illuminated by plane waves with tilted angles $\xi=45°$ ($r_1=5.6$ μm, $m_1=1.62$, $r_2=6.3$ μm, $m_2=1.505$, $r_3=7.0$ μm, $m_3=1.56$, $\lambda=633$ nm). The log scale is to the base 10.

sphere scattering and normal-cylinder scattering.

For the case of multilayered cylinder scattering, the GDSE is employed to the simulation of far-field scattered intensities for a doubly clad image-transmitting fiber obliquely illuminated by plane waves which is studied by Barabás [11] using Mie theory. Figure 5 depicts the far-field scattered intensity for a doubly clad image-transmitting fiber (solid line for TM polarization, dashed line for TE one) that correspond to Figs. 6(a) and 6(c) of Ref. [11]. The radii and refractive indices of the three-layered cylinder are $r_1=5.6$ μm, $m_1=1.62$, $r_2=6.3$ μm, $m_2=1.505$, $r_3=7.0$ μm, and $m_3=1.56$. The wavelength of incident waves is $\lambda=633$ nm, and the tilt angle is $\xi=45°$. The scattering intensity distribution in Fig. 5 is in agreement with that of Barabás.

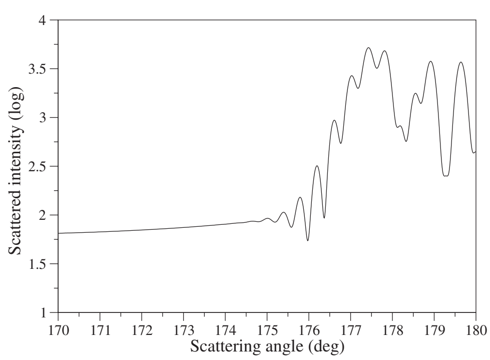
The DSE can be employed to simulate single-mode contributions easily and conveniently. Lock [20] employed the DSE to analyze the first-order rainbow produced in scattering of obliquely incident plane wave by homogeneous cylinders. The one-internal-reflection portion of the partial-wave scattering amplitudes is

$$B^{\mu}_{n,1}=-\tfrac{1}{2}(T^{21}_{\mu\mu}R^{121}_{\mu\mu}T^{12}_{\mu\mu}+T^{21}_{\mu\varepsilon}R^{121}_{\varepsilon\mu}T^{12}_{\mu\mu}+T^{21}_{\mu\mu}R^{121}_{\mu\varepsilon}T^{12}_{\varepsilon\mu}+T^{21}_{\mu\varepsilon}R^{121}_{\varepsilon\varepsilon}T^{12}_{\varepsilon\mu}), \qquad (147)$$

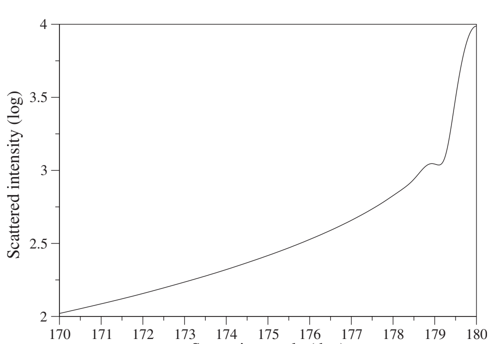
$$B^{\varepsilon}_{n,1}=-\tfrac{1}{2}(T^{21}_{\varepsilon\varepsilon}R^{121}_{\varepsilon\varepsilon}T^{12}_{\varepsilon\varepsilon}+T^{21}_{\varepsilon\mu}R^{121}_{\mu\varepsilon}T^{12}_{\varepsilon\varepsilon}+T^{21}_{\varepsilon\varepsilon}R^{121}_{\varepsilon\mu}T^{12}_{\mu\varepsilon}+T^{21}_{\varepsilon\mu}R^{121}_{\mu\mu}T^{12}_{\mu\varepsilon}), \qquad (148)$$

$$Q^{\mu}_{n,1}=-\tfrac{1}{2}(T^{21}_{\mu\mu}R^{121}_{\mu\mu}T^{12}_{\mu\varepsilon}+T^{21}_{\mu\varepsilon}R^{121}_{\varepsilon\mu}T^{12}_{\mu\varepsilon}+T^{21}_{\mu\mu}R^{121}_{\mu\varepsilon}T^{12}_{\varepsilon\varepsilon}+T^{21}_{\mu\varepsilon}R^{121}_{\varepsilon\varepsilon}T^{12}_{\varepsilon\varepsilon}). \qquad (149)$$

For the verification of our code, we show in Figs. 6(a) and 6(b) the one-internal-reflection scattered intensity obtained by our code in the same case as for Fig. 4 of Ref. [20] for a homogeneous cylinder with refractive index $m=1.484$, size parameter $2\pi a/\lambda=1000.0$ and $\xi=45.0°$ and 50.72°, respectively. The scattered intensity associated with the complete partial-wave scattering amplitudes of Eqs. (56), (59), and (63) is shown in Fig. 7 which correspond to of Fig. 6 of Ref. [20]. Because of symmetry only the part between 170° and 180° is given while the part between 180° and 190° is neglected. We find that the agreement of scattered intensity

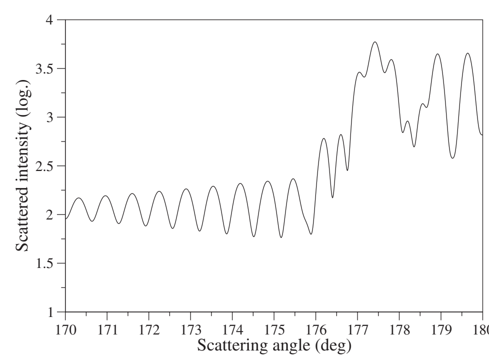


(a) $\xi=45.0°$

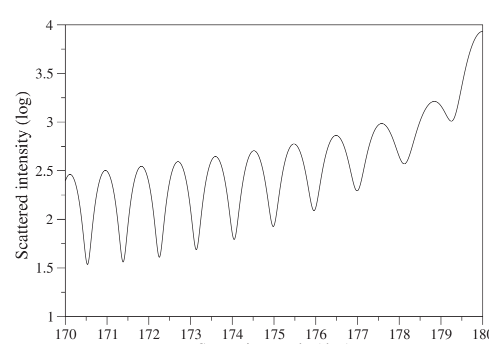


(b) $\xi=50.72°$

FIG. 6. One-internal-reflection portion of the scattered intensity for homogeneous cylinder with $m=1.484$ and $2\pi a/\lambda=1000.0$. The log scale is to the base 10.



(a) $\xi=45.0°$



(b) $\xi=50.72°$

FIG. 7. Scattered intensity for homogeneous cylinder with $m=1.484$ and $2\pi a/\lambda=1000.0$. The log scale is to the base 10.

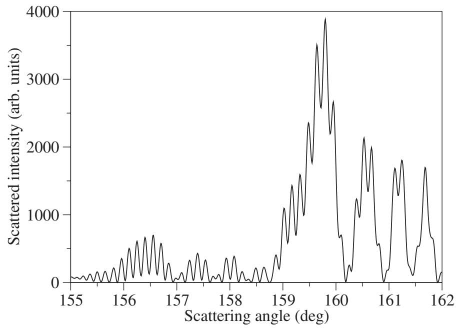


FIG. 8. Scattered intensities of twin first-order rainbows by a coated cylinder. The wavelength of the incident plane wave is $\lambda = 632.8$ nm. The refractive indices for the core and the coating are, respectively, 1.5 and 1.33. The corresponding diameters are $d_1 = 0.485$ mm and $d_2 = 0.505$ mm. The tilt angle is $\xi = 10°$.

distribution calculated by our code with that of Lock is excellent. It is necessary to point out that when all modes are taken into account, the results obtained by GDSE is identical to that of Mie theory.

DSE permits to identify the contributions of each mode of rays, and can be employed to the research on electromagnetic scattering by homogeneous and multilayered spheres, cylinders at normal incidence, and homogeneous cylinders at oblique incidence. But when cylinder is obliquely illuminated by plane waves, the DSE is not applicable. GDSE is just developed to such problem and it provides an efficient tool to study the contribution of different scattering modes. Multiple first-order rainbows formed by a multilayered cylinder is a typically example [4,17].

When a coated cylinder is illuminated by plane waves, twin first-order rainbows can be observed. Adler *et al.* [4,17] experimentally studied the twin primary rainbows when a water-coated glass rod is normally illuminated by plane waves. When the rod is obliquely illuminated by plane waves, we can also observe twin first-order rainbows, but the twin primary rainbows is more complicated than that for the normal incidence. Figure 8 depicts typical twin first-order rainbows produced by a water-coated glass rod obliquely illuminated by plane waves of wavelength $\lambda = 632.8$ nm. The refractive indices of the core and the coating are $m_1 = 1.5$ and $m_2 = 1.33$. The corresponding diameters are $d_1 = 0.485$ mm and $d_2 = 0.505$ mm. The refractive index of the surrounding is 1.0. The tilt angle is $\xi = 10°$. GDSE is employed to the research of such twin first-order rainbows. Figure 9 illustrates the rays of the first-order rainbows, denoted α and β, each of which undergoes one internal reflection on one of the two surfaces of the cylinder. Figure 10 depicts the polarization-preserving component of twin first-order rainbows produced by α ray and β ray with $B_{n,2}^{\alpha} = -\frac{1}{2} T_{\mu\mu}^{32} T_{\mu\mu}^{21} R_{\mu\mu}^{121} T_{\mu\mu}^{12} T_{\mu\mu}^{23}$ and $B_{n,2}^{\beta} = -\frac{1}{2} T_{\mu\mu}^{32} T_{\mu\mu}^{21} T_{\mu\mu}^{12} R_{\mu\mu}^{232} T_{\mu\mu}^{21} T_{\mu\mu}^{12} T_{\mu\mu}^{23}$. The scattered intensities produced by single rays of α ray or β ray are also given in Fig. 10. From the comparison of Fig. 8 and 10, we can find that the main bows between 155° and 158.5° and those between 159° and 162° are produced, respectively, by α ray and β ray, while the high frequency or ripple structure is due

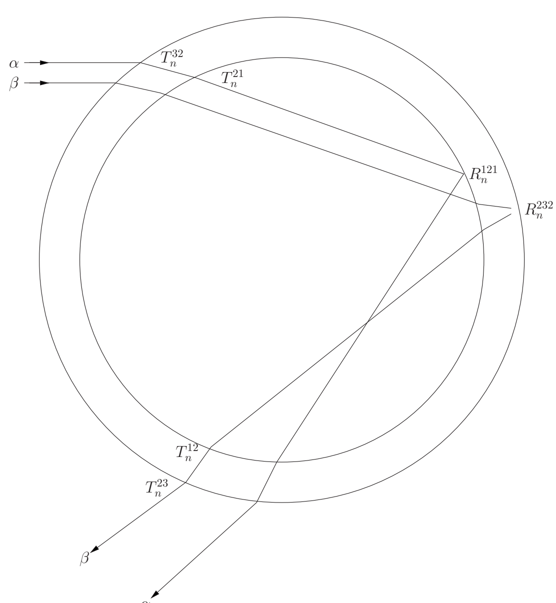


FIG. 9. Debye model for the formation of twin first-order rainbows of a coated cylinder.

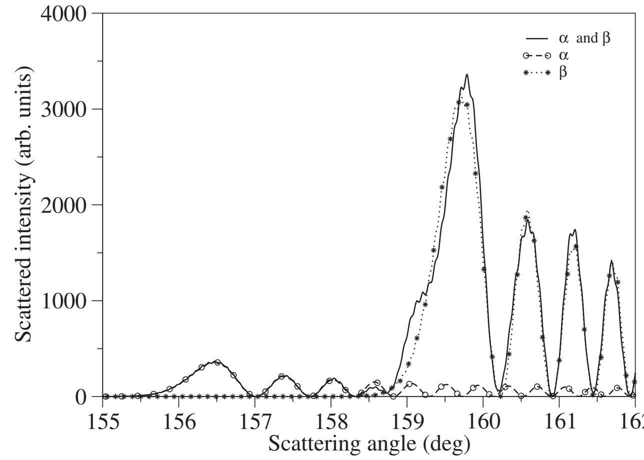


FIG. 10. Polarization-preserving component of twin first-order rainbows simulated by GDSE.

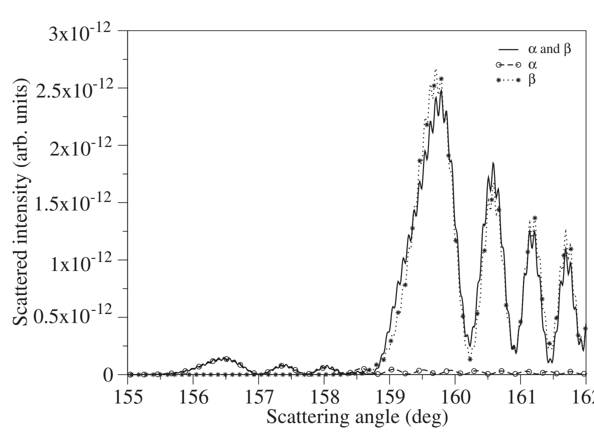


FIG. 11. Cross-polarization component of twin first-order rainbows simulated by GDSE with $B_{n,2}^{\alpha} = -\frac{1}{2} T_{\mu\mu}^{32} T_{\mu\mu}^{21} R_{\mu\varepsilon}^{121} T_{\varepsilon\mu}^{12} T_{\mu\mu}^{23}$ and $B_{n,2}^{\beta} = -\frac{1}{2} T_{\mu\mu}^{32} T_{\mu\mu}^{21} T_{\mu\mu}^{12} R_{\mu\varepsilon}^{232} T_{\varepsilon\mu}^{21} T_{\mu\mu}^{12} T_{\mu\mu}^{23}$.

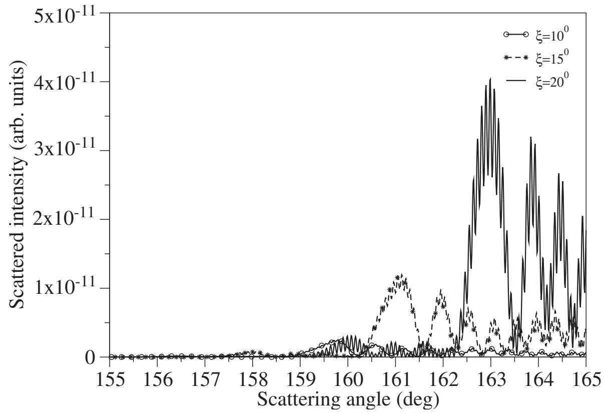


FIG. 12. Cross-polarization component of twin first-order rainbows vs tilt angle.

to the interference between these rays and the reflection rays.

When the rod is obliquely illuminated, there exists also the cross polarization. The cross-polarized intensity for twin primary rainbows produced by α ray with $B_{n,2}^{\alpha} = -\frac{1}{2}T_{\mu\mu}^{32}T_{\mu\mu}^{21}R_{\mu\varepsilon}^{121}T_{\varepsilon\mu}^{12}T_{\mu\mu}^{23}$ and β ray with $B_{n,2}^{\beta} = -\frac{1}{2}T_{\mu\mu}^{32}T_{\mu\mu}^{21}T_{\mu\mu}^{12}R_{\mu\varepsilon}^{232}T_{\varepsilon\mu}^{21}T_{\mu\mu}^{12}T_{\mu\mu}^{23}$ is considered as an example. Figure 11 depicts the cross-polarization component of first-order rainbows produced by such two rays. By comparison of Fig. 10 and 11. we can find that the polarization-preserving intensity and the cross-polarization one have same angular positions, but the magnitude of polarization-preserving intensity is much greater than that of the cross-polarization one.

It is necessary to point out that the cross-polarized intensity is very sensitive to tilt angles. Figure 12 gives the cross-polarized intensities of twin first-order rainbows with various tilt angles (10°, 15°, and 20°). When tilt angles become larger, both first-order rainbows move toward larger scattering angles, and the magnitude of intensities becomes more important.

VII. CONCLUSION

The generalized Debye series expansion (GDSE) for electromagnetic scattering is derived on the basis of the GDSE for acoustic scattering by using the relationship of boundary conditions between global scattering process and local scattering processes for the scattering of infinite multilayered cylinders obliquely illuminated by plane waves. The GDSE expresses the Mie scattering coefficients in a series of Fresnel coefficients, and gives the physical interpretation of scattering modes. An improved numerical algorithm is presented which permits the simulation for plane-wave scattering of cylinders of large size parameters and great number of layers. The formula and code are verified by comparison of our results with that presented in the literatures, and good agreements are obtained. GDSE is employed, as an example, to the study of the twin primary rainbows produced by coated cylinders obliquely illuminated by plane waves.

APPENDIX A: FRESNEL COEFFICIENTS

By solving corresponding equations using Cramer's rules, we can find all Fresnel coeffecicents as follows:

$$R_{\mu\mu}^{212} = \frac{1}{D_L'}\{U[H_n^{(2)}(\beta)]^2 - \kappa_1^2\kappa_2^2 U_{22}V_{12}\}, \tag{A1}$$

$$R_{\mu\mu}^{121} = \frac{1}{D_L'}\{U[H_n^{(1)}(\alpha)]^2 - \kappa_1^2\kappa_2^2 U_{11}V_{12}\}, \tag{A2}$$

$$R_{\varepsilon\varepsilon}^{212} = \frac{1}{D_L'}\{U[H_n^{(2)}(\beta)]^2 - \kappa_1^2\kappa_2^2 U_{12}V_{22}\}, \tag{A3}$$

$$R_{\varepsilon\varepsilon}^{121} = \frac{1}{D_L'}\{U[H_n^{(1)}(\alpha)]^2 - \kappa_2^2\kappa_1^2 U_{12}V_{11}\}, \tag{A4}$$

$$R_{\mu\varepsilon}^{212} = \kappa_1^2\kappa_2[H_n^{(2)}(\beta)]^2 G_1\frac{W_2}{D_L'}, \tag{A5}$$

$$R_{\mu\varepsilon}^{121} = \kappa_1\kappa_2^2[H_n^{(1)}(\alpha)]^2 G_2\frac{W_1}{D_L'}, \tag{A6}$$

$$R_{\varepsilon\mu}^{212} = -\kappa_2\kappa_1^2[H_n^{(2)}(\beta)]^2 G_1\frac{W_2}{D_L'}, \tag{A7}$$

$$R_{\varepsilon\mu}^{121} = -\kappa_1\kappa_2^2[H_n^{(1)}(\alpha)]^2 G_2\frac{W_1}{D_L'}, \tag{A8}$$

$$T_{\mu\mu}^{21} = \kappa_1\kappa_2^4\frac{V_{12}W_2}{D_L'}, \tag{A9}$$

$$T_{\mu\mu}^{12} = \kappa_2\kappa_1^4\frac{W_1V_{12}}{D_L'}, \tag{A10}$$

$$T_{\varepsilon\varepsilon}^{21} = \kappa_1\kappa_2^4\frac{U_{12}W_2}{D_L'}, \tag{A11}$$

$$T_{\varepsilon\varepsilon}^{12} = \kappa_1^4\kappa_2\frac{U_{12}W_1}{D_L'}, \tag{A12}$$

$$T_{\mu\varepsilon}^{21} = \kappa_2^3 H_n^{(2)}(\beta)H_n^{(1)}(\alpha)G_1\frac{W_2}{D_L'}, \tag{A13}$$

$$T_{\mu\varepsilon}^{12} = \kappa_1^3 H_n^{(1)}(\alpha)H_n^{(2)}(\beta)G_2\frac{W_1}{D_L'}, \tag{A14}$$

$$T_{\varepsilon\mu}^{21} = -\kappa_2^3 H_n^{(1)}(\alpha)H_n^{(2)}(\beta)G_2\frac{W_2}{D_L'}, \tag{A15}$$

$$T_{\varepsilon\mu}^{12} = -\kappa_1^3 H_n^{(2)}(\beta)H_n^{(1)}(\alpha)G_1\frac{W_1}{D_L'}, \tag{A16}$$

$$D_L' = \kappa_1^2\kappa_2^2 U_{12}V_{12} + G_1G_2[H_n^{(1)}(\alpha)H_n^{(2)}(\beta)]^2, \tag{A17}$$

$$U = -G_1G_2H_n^{(1)}(\alpha)H_n^{(2)}(\alpha), \tag{A18}$$

$$U_{ij} = \frac{m_1}{m_2}\kappa_2 H_n^{(i)}(\alpha) H_n^{(j)\prime}(\beta) - \frac{m_2}{m_1}\kappa_1 H_n^{(i)\prime}(\alpha) H_n^{(j)}(\beta), \tag{A19}$$

$$V_{ij} = \kappa_2 H_n^{(i)}(\alpha) H_n^{(j)\prime}(\beta) - \kappa_1 H_n^{(j)}(\beta) H_n^{(i)\prime}(\alpha), \tag{A20}$$

$$W_i = H_n^{(2)\prime}(L) H_n^{(1)}(L) - H_n^{(2)}(L) H_n^{(1)\prime}(L), \tag{A21}$$

$$G_i = \frac{inh(\kappa_1^2 - \kappa_2^2)}{k_i a}, \tag{A22}$$

$$L = \begin{cases} \beta & i = 1, \\ \alpha & i = 2. \end{cases} \tag{A23}$$

APPENDIX B: REFORMED FRESNEL COEFFICIENTS USING LOGARITHMIC DERIVATIVES

Using logarithmic derivatives, we can obtain reformed Fresnel coefficients as follows:

$$R_{\mu\mu}^{212} = \frac{H_n^{(2)}(\alpha)}{H_n^{(1)}(\alpha)} \frac{n^2h^2(\kappa_1^2 - \kappa_2^2)^2 - \kappa_1^2\kappa_2^2 k_0^2 a^2 U_{22} V_{21}}{D_L''}, \tag{B1}$$

$$R_{\mu\mu}^{121} = \frac{H_n^{(1)}(\beta)}{H_n^{(2)}(\beta)} \frac{n^2h^2(\kappa_1^2 - \kappa_2^2)^2 - k_0^2 a^2 \kappa_1^2\kappa_2^2 V_{21} U_{11}}{D_L''}, \tag{B2}$$

$$R_{\varepsilon\varepsilon}^{212} = \frac{H_n^{(2)}(\alpha)}{H_n^{(1)}(\alpha)} \frac{n^2h^2(\kappa_1^2 - \kappa_2^2)^2 - k_0^2 a^2 \kappa_1^2\kappa_2^2 U_{21} V_{22}}{D_L''}, \tag{B3}$$

$$R_{\varepsilon\varepsilon}^{121} = \frac{H_n^{(1)}(\beta)}{H_n^{(2)}(\beta)} \frac{n^2h^2(\kappa_1^2 - \kappa_2^2)^2 - k_0^2 a^2 \kappa_1^2\kappa_2^2 U_{21} V_{11}}{D_L''}, \tag{B4}$$

$$R_{\mu\varepsilon}^{212} = k_2 ainh\kappa_1^2\kappa_2(\kappa_1^2 - \kappa_2^2)\frac{H_n^{(2)}(\alpha)}{H_n^{(1)}(\alpha)}\frac{W_2}{D_L''}, \tag{B5}$$

$$R_{\mu\varepsilon}^{121} = k_1 ainh\kappa_1\kappa_2^2(\kappa_1^2 - \kappa_2^2)\frac{H_n^{(1)}(\beta)}{H_n^{(2)}(\beta)}\frac{W_1}{D_L''}, \tag{B6}$$

$$R_{\varepsilon\mu}^{212} = k_2 ainh\kappa_2\kappa_1^2(\kappa_2^2 - \kappa_1^2)\frac{H_n^{(2)}(\alpha)}{H_n^{(1)}(\alpha)}\frac{W_2}{D_L''}, \tag{B7}$$

$$R_{\varepsilon\mu}^{121} = k_1 ainh\kappa_1\kappa_2^2(\kappa_2^2 - \kappa_1^2)\frac{H_n^{(1)}(\beta)}{H_n^{(2)}(\beta)}\frac{W_1}{D_L''}, \tag{B8}$$

$$T_{\mu\mu}^{21} = k_1 k_2 a^2 \kappa_1 \kappa_2^4 \frac{H_n^{(2)}(\alpha)}{H_n^{(2)}(\beta)}\frac{V_{21}W_2}{D_L''}, \tag{B9}$$

$$T_{\mu\mu}^{12} = \kappa_2\kappa_1^4 k_1 k_2 a^2 \frac{H_n^{(1)}(\beta)}{H_n^{(1)}(\alpha)}\frac{W_1 V_{21}}{D_L''}, \tag{B10}$$

$$T_{\varepsilon\varepsilon}^{21} = \kappa_1\kappa_2^4 k_0^2 a^2 \frac{H_n^{(2)}(\alpha)}{H_n^{(2)}(\beta)}\frac{U_{22}W_2}{D_L''}, \tag{B11}$$

$$T_{\varepsilon\varepsilon}^{12} = k_1 k_2 a^2 \kappa_1^4 \kappa_2 \frac{H_n^{(1)}(\beta)}{H_n^{(1)}(\alpha)}\frac{V_{21}W_1}{D_L''}, \tag{B12}$$

$$T_{\mu\varepsilon}^{21} = k_2 ainh(\kappa_1^2 - \kappa_2^2)\kappa_2^3 \frac{H_n^{(2)}(\alpha)}{H_n^{(2)}(\beta)}\frac{W_2}{D_L''}, \tag{B13}$$

$$T_{\mu\varepsilon}^{12} = k_1 ainh\kappa_1^3(\kappa_1^2 - \kappa_2^2)\frac{H_n^{(1)}(\beta)}{H_n^{(1)}(\alpha)}\frac{W_1}{D_L''}, \tag{B14}$$

$$T_{\varepsilon\mu}^{21} = k_1 ainh\kappa_2^3(\kappa_2^2 - \kappa_1^2)\frac{H_n^{(2)}(\alpha)}{H_n^{(2)}(\beta)}\frac{W_2}{D_L''}, \tag{B15}$$

$$T_{\varepsilon\mu}^{12} = k_2 ainh\kappa_1^3(\kappa_2^2 - \kappa_1^2)\frac{H_n^{(1)}(\beta)}{H_n^{(1)}(\alpha)}\frac{W_1}{D_L''}, \tag{B16}$$

$$D_L'' = \kappa_1^2\kappa_2^2 k_0^2 a^2 U_{21} V_{21} - n^2h^2(\kappa_1^2 - \kappa_2^2)^2, \tag{B17}$$

$$U_{ij} = m_1^2 \kappa_2 D_n^{(i)}(\beta) - m_2^2 \kappa_1 D_n^{(j)}(\alpha), \tag{B18}$$

$$V_{ij} = \kappa_2 D_n^{(i)}(\beta) - \kappa_1 D_n^{(j)}(\alpha), \tag{B19}$$

$$W_i = D_n^{(2)}(L) - D_n^{(1)}(L). \tag{B20}$$

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